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P. D. Spanos, L. B. Ryon Professor of Engineering

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Rice University
Office of Sponsored Research
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Houston, TX 77252

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13. ABSTRACT *(Maximum 200 words)*

Based on the support of the aforementioned grant, considerable advances were made in several structural mechanics problems with stochasticity. First, a new method was developed for the solution of problems involving material variability. The Karhunen-Loeve expansion was used for this purpose. Further, a problem involving a plate with curved boundary was solved. Also methods unifying the theory of stochastic and deterministic finite elements were developed. In parallel, a method for more efficient Monte-Carlo simulation was developed. Finally, some work in the area of stochastic finite difference was done. Two research assistants have participated in the program; the one under support from discretionary funds of Rice University. Five research papers with acknowledgment of the support from AFOSR have been produced.

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AFOSR Grant 91-0004
"A Novel Approach For Structural Mechanics Problems With Stochasticity"

by P. D. Spanos
Mechanical Engineering
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Houston, Texas

Based on the support provided by the aforementioned grant, considerable advances were made in several structural mechanics problems with stochasticity.

A new method for the solution of problems involving material variability was developed. The material variability was modeled as a stochastic process. The Karhunen-Loeve expansion of random processes was used to represent the material variability in a computationally expeditious manner. The well-known deterministic finite element method has been employed to discretize the differential equation which governs the nodal response random variables. An related spectral expansion of these random variables was adopted in terms of the basis in the space of second nodal random variables. This method yielded a representation of the response surface in terms of the polynomial chaos. The coefficient in this representation was such that it involved enough information about the process so that one could reproduce its probability distribution function. The method has been applied to a plain-stress problem which involves a curved geometrical boundary. The representation of the random field over the curved domain was accomplished by solving the related integral equation using a Galerkin formulation. Interestingly, the result of the representation is independent of the mass size which was employed, and converged quite rapidly as the number of terms in the Karhunen-Loeve expansion increased. Even more encouraging was the fact that the analytical results were found in extremely good agreement with data produced by a Monte-Carlo study of the problem. The findings of this research effort have been summarized in the paper:

> "Stochastic Finite Element Analysis With Curved Boundaries," by R. G. Ghanem and P. D. Spanos, Proceedings of the Sixth International Conference on Application of Statistics and probability in Civil Engineering, Mexico City, Mexico, 1991, June 7-9, 1991, pp. 158-165.

An effort was undertaken to develop a formulation of stochastic concepts towards a unification of various finite element techniques. Specifically, methods for the solution of differential equations with random processes and coefficients have been addressed. The method which was advocated treats the random aspects of the problem as an added dimension. In this

manner, a formulation for the stochastic finite element method has been derived which can be construed as a natural extension of the deterministic finite element method. Finite element representation along the random dimension was achieved by two spectral expansions. One of them was used to represent the coefficients of the differential equation which modeled the random material property, and the other one was used to represent the random solution process. The new concepts were implemented by using a number of computational models as simple engineering systems. The new method indeed involves generalizing the concepts of finite element approximation to abstract spaces of which the usual Euclidian space is a special case. The deterministic case can be regarded as a digression of this formalism to the particular case where the space of elementary events involves only a single element, and where the probability density function induced on the associated α-algebra is the uniform distribution. The findings of these research efforts have been summarized in the paper:

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Another numerical method for dealing with problems of stochastic mechanics has been pursued and it involves a small number of random parameters. This method is analogous to the Monte-Carlo simulation method, but it is more efficient. In fact, the method treats a stochastic problem as an ensemble of deterministic ones. After solving a number of deterministic problems, statistical analysis is performed to deduce the necessary parameters characterizing the random nature of the solution. Of course, this method is often the only option which is available to solve complicated stochastic problems. However, indiscriminate use of the method is not advocated due to the significant computational cost. The new method has been used for the determination of the eigenvalues of a beam bending problem with random parameters. The beam is assumed to be clamped-clamped of unit length, and its rigidity is a truncated normal process with mean equal to one and with exponential auto-correlation function. This kind of problem is quite difficult to be treated either analytically or by other numerical methods. In fact, the available algorithms can be applied in the case of very small randomness and they are quite costly computationally. The new method has been used to determine the first three eigenvalues of this problem. It was found that the analytical results are in very good agreement with the results obtained by numerical simulations of this problem. Further, it has been found that the new method can be applied to a wide class of problems dealing with random variables and stochastic processes. The findings of this research effort have been summarized in the paper:

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Another approach to dealing with problems of the numerical solution of stochastic mechanics problems has been pursued. Specifically, the Finite Difference Method for discretization of stochastic continuous media has been addressed. The Neumann expansion and perturbation methods used for solving the systems of the associated algebraic equations have been studied. Further, work has been done into the dependence on the mass size of the discretization. It has been found that a mixed formulation which involves both strain and stresses as independent variables improves the performance of the method and reduces the dependence on the mass size. The method has been used to study the behavior of a beam which is subjected to deterministic load and involves bending rigidity which is a normal random process. It has been found that the mixed formulation includes the convergence of the Neumann expansion, and it minimizes its dependence on the mass size. This is particularly true with regards to the variability of the displacement, even when the coefficient of variation of the bending rigidity is quite large. The findings of this research effort have been summarized in the paper:

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A class of Galerkin-type projection procedures applied for random media for stochastic mechanics problems has been developed. The use of a space of simple random variables as projective has been adopted. Further, it has been shown that in this case this procedure is equivalent to a generalization of sampling technique. Then, the optimality of this indirect sampling has been investigated. It has been shown that this method is a generalized stratified sampling method. That is, the approximation of the problem from the space of sample random variables produces an error with mean and conditional expectation, given sigma-algebra induced by this partition, equal zero. It has been shown that this error has zero estimate from the set of indicator functions of given stratification. Some stochastic mechanics problems have been considered utilizing the proposed method. It has been shown that the Loeve-Karhunen expansion is a versatile tool for the approximation of a stochastic field through a set of random variables. This

expansion is optimal and, therefore, a small number of random variables can approximate them adequately. Several examples have demonstrated that the proposed method can be applied for treating a wide class of problems dealing with random variables and stochastic processes. The findings of this research effort have been summarized in the paper:

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In terms of personnel, it is stated that in addition to the supervising professor, two research assistants were involved in the projects. One was supported exclusively by the research grant from AFOSR, and another was jointly supported by this AFOSR grant and discretionary funds from Rice University. The two research assistants and the supervising professor formed a very coherent group which worked with enthusiasm in this challenging technical field.

APPENDIX

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Proceedings of the Sixth International Conference on Applications of Statistics and Probability in Civil Engineering, 1991, Mexico City, Mexico

1)

Stochastic Finite Element Analysis with Curved Boundaries

R.G. Ghanem * P D. Spanos †

Abstract

An original method for the solution of problems involving material variability is proposed. The material property is modeled as a stochastic process. The Karhunen-Loeve expansion is used to represent this process in a computationally expedient manner. The standard deterministic finite element method is employed to discretize the differential equations governing the nodal response random variables. A spectral expansion of these random variables is then adopted in terms of a basis in the space of second order random variables. The method yields an expression for the response surface in terms of the Polynomial Chaoses. The coefficients in the expansion are such that they involve enough information about the response process so as to be possible to reproduce its probability distribution function. The method is applied to a plane-stress problem involving a curved geometrical boundary. Representation of the random field over the curved domain is accomplished by solving the related integral equation using a *Galerkin formulation*. The resulting representation is independent of the mesh size employed and converges rapidly as the number of terms in the Karhunen-Loeve expansion is increased.

1 Introduction

The analysis of engineering systems with uncertain properties has witnessed a considerable resurgence in recent years, in particular with relation to systems involving many degrees of freedom and requiring recourse to the finite element method for their analysis and design. A comprehensive treatment of this class of problems can be accomplished by breaking down the complexity into two separate issues. The first one consists of adequately representing the uncertainty in the system properties for implementation within a computational framework. The Karhunen-Loeve expansion is used in this paper as an optimal such representation. It effectively replaces the random process by a set of uncorrelated random variables while delegating the corresponding *spatial dependence to a set of* deterministic functions. The second issue involved in the solution process is obtaining a representation for the response process. This task is accomplished by first identifying a complete basis for such a representation, and then by formulating the problem in such a way that the coefficients of this basis in the representation can be numerically computed. In this paper, earlier formulations of the ideas presented above (Spanos and Ghanem, 1989, 1990, 1991) are extended to deal with situations where the geometry of the domain under consideration is irregular, involving curved boundaries. The efficiency of the proposed method in treating such problems is demonstrated by its application to the analysis of a curved thin plate. In addressing this problem, it is reminded that the ultimate goal of a stochastic finite element analysis is the calculation of certain statistics of the response process. These statistics can be in the form of either statistical moments, or probability distribution function, or some other measure of the reliability of the system. As a first step in the solution procedure, the variational formulation of the finite element method is used to obtain a spatially discrete form of the problem. Following that, the Polynomial Chaos expansion is used to derive a representation of the response process. Statistical moments and probability distribution functions can then obtained.

*Civil Engineering Department, State University of New York, Buffalo, NY 14260.
†L.B. Ryon Chair in Engineering, Rice University, Houston, TX, 77251.

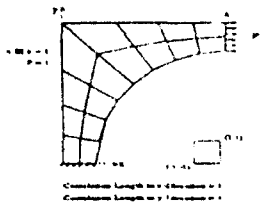

Figure 1: Plate with Random Rigidity, Exponential Covariance Model.

2 Finite Element Formulation

Consider th thin plate shown in Figure (1). Its modulus of elasticity is assumed to be the realization of a two-dimensional Gaussian random process with known mean value $\bar{E}$ and known covariance function $C(\mathbf{x}_1, \mathbf{x}_2)$. Further, it is assumed that the external excitation is deterministic and of unit magnitude. Let the domain A of the plate be discretized into N four-noded quadrilateral finite elements, each element having eight degrees of freedom. The strain energy V^e stored in each element A^e can be expressed as

$$V^e = \frac{1}{2} \int_{A^e} \sigma^T(\mathbf{x})\, \epsilon(\mathbf{x})\, dA^e , \tag{1}$$

where dA^e is a differential element in A^e. Further, $\sigma(\mathbf{x})$ and $\epsilon(\mathbf{x})$ denote the stress and the strain vectors respectively, as a function of the location $\mathbf{x}$ within each element. Assuming linear elastic material behavior, the stress may be expressed in terms of the strain as

$$\sigma = \mathbf{D}^e \epsilon \tag{2}$$

where $\mathbf{D}^e$ is the matrix of constitutive relations. Here σ and ϵ are the vectors of stress and strain as given by the equation

$$\sigma^T = [\, \sigma_{x_1} \; \sigma_{x_2} \; \tau_{x_1 x_2} \,] \tag{3}$$

$$\epsilon^T = [\, \epsilon_{x_1} \; \epsilon_{x_2} \; \gamma_{x_1 x_2} \,] , \tag{4}$$

where σ_{x_i} is the stress along direction x_i and ϵ_{x_i} is the strain along that same direction. For the plane stress problem considered herein, $\mathbf{D}^e$ is given by the equation

$$\mathbf{D}^e = \frac{E^e(\mathbf{x})}{1-\mu_e^2} \begin{bmatrix} 1 & \mu_e & 0 \\ \mu_e & 1 & 0 \\ 0 & 0 & \frac{(1-\mu_e)}{2} \end{bmatrix} = E^e(\mathbf{x})\, \mathbf{P}^e , \tag{5}$$

where $\mathbf{P}^e$ is a deterministic matrix, μ_e is the elemental Poisson ratio, and $E^e(\mathbf{x})$ is the elemental modulus of elasticity. The two dimensional displacement vector $\mathbf{u}(\mathbf{x})$ representing the longitudinal and transverse displacements within each element may be expressed in terms of the nodal displacements of the element in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{H}^e(r_1, r_2)\, \mathbf{U}^e , \tag{6}$$

where $\mathbf{H}^e(r_1, r_2)$ is the local interpolation matrix, $\mathbf{U}^e$ is the random nodal response vector, and r_1 and r_2 are local coordinates over the element. Substituting equations (5) and (6) into equation (1) gives

$$V^e = \frac{1}{2} \int_{A^e} E^e(\mathbf{x})\, \epsilon^T(\mathbf{x})\, \mathbf{P}^e\, \epsilon(\mathbf{x})\, dA^e . \tag{7}$$

The strain within an element is related to the displacements, longitudinal and transverse, through the relation

$$\epsilon(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 \\ 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{bmatrix} \mathbf{u}(\mathbf{x}) . \tag{8}$$

Using equation (6), equation (7) is rewritten as

$$\epsilon(\mathbf{x}) = \mathbf{B}^e \mathbf{U}^e , \tag{9}$$

where matrix B involves derivatives of the interpolation matrix H. Substituting equation (9) back into equation (7) and performing a coordinate transformation, leads to

$$V^e = \frac{1}{2} \mathbf{U}^{eT} \int_0^1 \int_0^1 E(r_1, r_2)\, \mathbf{B}^{eT}(r_1, r_2)\, \mathbf{P}^e\, \mathbf{B}^e(r_1, r_2)\, |\mathbf{J}^e| dr_1\, dr_2\, \mathbf{U}^e , \tag{10}$$

where $|\mathbf{J}^e|$ denotes the determinant of the Jacobian of the transformation that maps an arbitrary element (e) onto the four-noded square with sides equal to one. The total strain energy V is obtained by summing the contributions from all the elements. This procedure gives

$$V = \frac{1}{2} \sum_{e=1}^{N} \mathbf{U}^{eT} \int_0^1 \int_0^1 E(r_1, r_2)\, \mathbf{B}^{eT}(r_1, r_2)\, \mathbf{P}^e\, \mathbf{B}^e(r_1, r_2)\, |\mathbf{J}^e|\, dr_1\, dr_2\, \mathbf{U}^e \tag{11}$$

The local representation of the response is related to the global representation through the following transformation

$$\mathbf{U}^e = \mathbf{C}^e \mathbf{U} , \tag{12}$$

where $\mathbf{C}^e$ is a rectangular permutation matrix of zeros and ones reflecting the connectivity of the elements and the topology of the mesh. Using equation (12), the following expression for the total energy stored in the system is obtained

$$V = \frac{1}{2} \mathbf{U}^T \mathbf{K}\, \mathbf{U} . \tag{13}$$

Before evaluating the integrals in equation (11), the random process representing the modulus of elasticity of the plate must be replaced by a representation which somehow decouples its spatial dependence from its random dependence. The Karhunen-Loeve expansion (Spanos and Ghanem, 1988) is used herein as an optimal such representation. Accordingly, $E(r_1, r_2)$ is replaced by

$$E(r_1, r_2) = \sum_{k=1}^{M} \sqrt{\lambda_k} \xi_k f_k(r_1, r_2) , \tag{14}$$

where λ_k and $f_k(r_1, r_2)$ are, respectively, the eigenvalues and eigenfunctions associated with the covariance function of the random process. The next stage in the computations, therefore, involves solving the integral eigenvalue problem associated with the covariance kernel. That is,

$$\lambda_n f_n(x_1, y_1) = \int_A C(x_1, y_1; x_2, y_2)\, f_n(x_2, y_2)\, dx_2\, dy_2 . \tag{15}$$

The kernel used in this paper is defined by the equation

$$C(x_1, x_2; y_1, y_2) = e^{-|x_1 - y_1|/b_1 - |x_2 - y_2|/b_2} , \tag{16}$$

where b_1 and b_2 are the correlation distances in the x_1 and x_2 directions respectively.

3 Numerical Solution of the Integral Equation

Subdividing the domain A of the plate into N finite elements A^e, equation (15) becomes

$$\lambda_n f_n(x_1, y_1) = \sum_{e=1}^{N} \int_{A^e} C(x_1, y_1; x_2, y_2) f_n(x_2, y_2) \, dA^e . \tag{17}$$

Interpolating for the value of the unknown function within an element in terms of its nodal values results in the following expression

$$f_n(x, y) = \mathrm{H}^e(r_1, r_2) \, \mathrm{f}_n^e , \tag{18}$$

where $\mathrm{H}^e(r_1, r_2)$ is the interpolation matrix in terms of local coordinates r_1 and r_2, and f_n^e is the vector of nodal values for the unknown function associated with element (e). For this particular problem, bilinear interpolation is used over four-noded quadrilateral element. The matrix $\mathrm{H}^e(r_1, r_2)$ is then given by the equation

$$\mathrm{H}^e(r_1, r_2) = \frac{1}{4} \left[\begin{array}{cc} (1 - r_1)(1 - r_2) & (1 + r_1)(1 - r_2) \\ (1 + r_1)(1 + r_2) & (1 - r_1)(1 + r_2) \end{array} \right] . \tag{19}$$

Substituting equation (18) and performing a transformation from global to local coordinates, equation (17) becomes

$$\lambda_n f_n(x_1, y_1) = \sum_{e=1}^{N} \int_{A^e} C(x_1, y_1; x_2, y_2) \, \mathrm{H}^e(r_1, r_2) \, |J^e| \, dA^e \, \mathrm{f}_n^e , \tag{20}$$

where $|J^e|$ is the Jacobian of the coordinate transformation. A system of algebraic equations is obtained from equation (28) by requiring the corresponding error to be orthogonal to all the interpolation functions used. That is,

$$\mathbf{C} \, \mathbf{D} = \mathbf{\Lambda} \, \mathbf{B} \, \mathbf{D} , \tag{21}$$

where now the j^{th} column of $\mathbf{D}$ is the j^{th} eigenfunction calculated at the nodal points and

$$\Lambda_{ij} = \delta_{ij} \, \lambda_i . \tag{22}$$

Matrices $\mathbf{C}$ and $\mathbf{B}$ are obtained by assembling matrices $\mathbf{C}_{ef}$ and $\mathbf{B}_{ef}$ where

$$\mathbf{C}_{ef} = \int_{A^e} \int_{A^f} C(x_1, y_1; x_2, y_2) \, \mathrm{H}^{eT}(r_1, r_2) \, \mathrm{H}^f(r_1, r_2) \, dA^e \, dA^f , \tag{23}$$

and

$$\mathrm{B}_{ef} = \int_{A^e} C(x_1, y_1; x_2, y_2) \, \mathrm{H}^{eT}(r_1, r_2) \, \mathrm{H}^f(r_1, r_2) \, dA^e , \tag{24}$$

with (x_1, y_1) and (r_1, r_2) denoting the global and local coordinates of a point in A^e respectively. The assembly procedure just mentioned consists of combining entries corresponding to the same node (Akin, 1982).

4 Spectral Stochastic Finite Elements

The Karhunen-Loeve expansion for the modulus of elasticity may be substituted into equation (11) to transform equation (13) into

$$V = \frac{1}{2} \mathbf{U}^T \sum_{k=1}^{M} \xi_k \, \mathbf{K}^{(k)} \, \mathbf{U} \tag{25}$$

The integrations involved in equation (25) may be performed either analytically or using some numerical quadrature scheme. The work performed by the externally applied forces is

$$V' = \sum_{e=1}^{N} \int_{A^e} \mathrm{u}^{e\,T}(\mathrm{x})\, \mathrm{f}(\mathrm{x})\, dA^e ,$$

$$= \mathrm{U}^T \sum_{k=1}^{M} \mathrm{C}^{eT} \int_{A^e} \mathrm{H}^{e\,T}(\mathrm{x})\, \mathrm{f}(\mathrm{x})\, dA^e = \mathrm{U}^T \mathrm{f} , \tag{26}$$

where

$$\mathrm{f} = \sum_{e=1}^{N} \mathrm{C}^{eT} \int_{A^e} \mathrm{H}^{e}(\mathrm{x})\, \mathrm{f}(\mathrm{x})\, dA^e . \tag{27}$$

Minimizing the total potential energy $(V - V')$ with respect to U, leads to the equation

$$\left[\mathrm{K}^{(0)} + \sum_{k=1}^{M} \xi_k \mathrm{K}^{(k)} \right] \mathrm{U} = \mathrm{f} . \tag{28}$$

The response vector U is expanded along the basis given by the Polynomial Chaoses (Ghanem and Spanos, 1990) as,

$$\mathrm{U} = \sum_{i=0}^{P} \mathrm{c}_i\, \Psi_i[\{\xi_l\}] . \tag{29}$$

Substituting equation (29) into equation (28), and requiring the error resulting from truncating the series at the P^{th} term to be orthogonal to the $P+1$ Polynomial Chaoses $\Psi_i[\{\xi_l\}]_{i=0}^{P}$, results in a system of linear algebraic equations of the form

$$\sum_{i=0}^{P} \left[\sum_{k=0}^{M} <\xi_k \Psi_i[\{\xi_l\}]\Psi_j[\{\xi_l\}]> \mathrm{K}^{(k)} \right] \mathrm{c}_i = <\Psi_j[\{\xi_l\}]\, \mathrm{f}> , \quad j = 0 \ldots P , \tag{30}$$

which can be solved for the coefficients c_i.

5 Numerical Results

A curved plate is shown in Figure (1). The curved side is a ninety degree arc of a circle of unit radius. The length of the straight edges is equal to 1.25. The standard deviation of the longitudinal displacement at node A, using two terms in the K-L expansion for the material stochasticity, is shown in Figure (2). It is plotted against the standard deviation of the modulus of elasticity. The results corresponding to four terms in the K-L expansion are shown in Figure (3). Note the excellent convergence. The probability distribution functions corresponding to one of the response variable at node A are depicted in Figures (4)-(7). Results corresponding to two and four terms in the Karhunen-Loeve expansion and up to third order Polynomial Chaos are shown.

The two dimensional process representing the modulus of elasticity of the plate is simulated in such a way as to accommodate the non-uniform spatial distribution of the nodal points. The issue is addressed by using the Karhunen-Loeve expansion to simulate a truly continuous random field. To this end, the eigenvalues and eigenfunctions of the covariance kernel are computed as described in the previous section. The random field is then simulated using equation (14) with the number of terms equal to the number of nodes in the system. The orthogonal random variables appearing in that equation are obtained as pseudorandom computer generated uncorrelated variates, with zero mean and unit variance. The resulting simulated random field is not as sensitive to the mesh size and

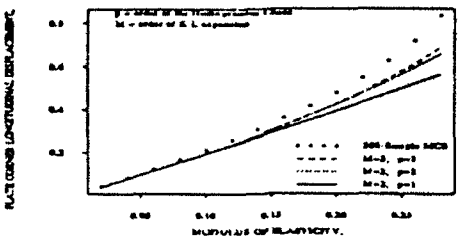

Figure 2: Normalized Standard Deviation of Longitudinal Displacement at Corner A of the Curved Plate, versus Standard Deviation of the Modulus of Elasticity; Two Terms in the K-L Expansion; Exponential Covariance; $\sigma_{max} = 19.4$.

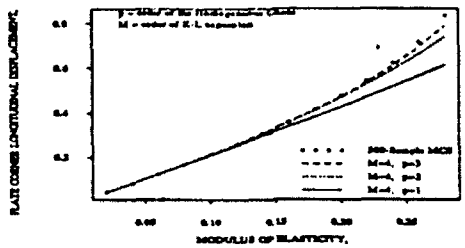

Figure 3: Normalized Standard Deviation of Longitudinal Displacement at Corner A of the Curved Plate, versus Standard Deviation of the Modulus of Elasticity; Four Terms in the K-L Expansion; Exponential Covariance; $\sigma_{max} = 19.4$.

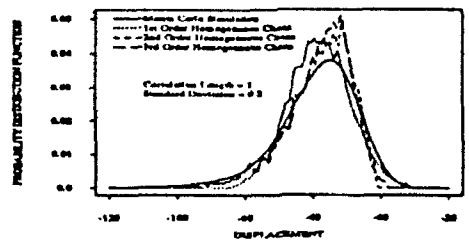

Figure 4: Longitudinal Displacement at the Free End of the Curved Plate; Probability Density Function Using 30,000-Samples MSC, and Using Third Order Homogeneous Chaos; Two Terms in the K-L Expansion; Exponential Covariance.

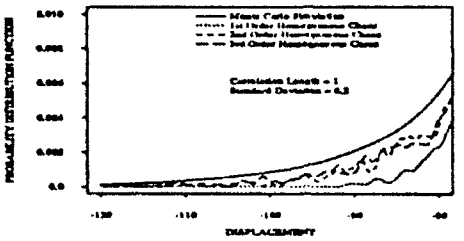

Figure 5: Longitudinal Displacement at the Free End of the Curved Plate; Tail of the Probability Density Function Using 30,000-Sample MSC, and Using Third Order Homogeneous Chaos; Two Terms in the K-L Expansion; Exponential Covariance.

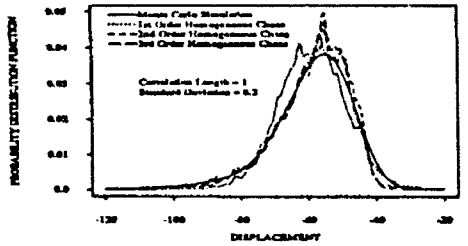

Figure 6: Longitudinal Displacement at the Free End of the Curved Plate; Probability Density Function Using 30,000-Sample MSC, and Using Third Order Homogeneous Chaos; Four Terms in the K-L Expansion; Exponential Covariance.

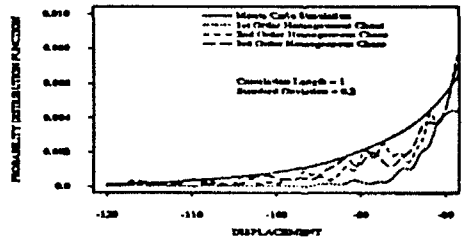

Figure 7: Longitudinal Displacement at the Free End of the Curved Plate; Tail of the Probability Density Function Using 30,000-Sample MSC, and Using Third Order Homogeneous Chaos; Four Terms in the K-L Expansion; Exponential Covariance.

nodal point distribution as the field obtained using more conventional procedures. The results from using the Monte Carlo simulation method are superimposed on the same plot as the analytical results. Observe the good agreement between the analytical and the simulated results even for large values of the coefficient of variation.

6 Acknowledgement

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CHAPTER 5

2) # A SPECTRAL FORMULATION OF STOCHASTIC FINITE ELEMENTS

Roger G. Ghanem and Pol D. Spanos

A SPECTRAL FORMULATION OF STOCHASTIC FINITE ELEMENTS

Roger G. Ghanem

Department of Civil Engineering
State University of New York at Buffalo
Buffalo, NY 14260.

Pol D. Spanos

L.B. Ryon Professor of Engineering
School of Engineering
Rice University
Houston, TX. 77251.

Abstract

A method for the solution of differential equations with random processes as coefficients is developed. The method relies on viewing the random aspect of the problem as an added dimension, and on treating random variables and processes as functions defined over that dimension. This way, a formulation for the stochastic finite element method is developed which is a natural extension of the deterministic finite element method. Finite element representation along the random dimension is achieved via two spectral expansions. One of them is used to represent the coefficients of the differential equation which model the random material properties, the other is used to represent the random solution process.

1. INTRODUCTION

Although the first quantitative ideas of probability theory took form as far back as the seventeenth century, at the gambling tables, with Pascal and Fermat, it was not until well into this century that the body of knowledge known as probability became a mathematical discipline. In the interim, some of the greatest scientific minds, including Gauss, Laplace, and Poincare, as well as many others, were laying the foundation for the theory by compiling observations and related theorems on the chance occurrence of events. Several attempts to baptize the accumulated ideas as a branch of mathematics were vehemently opposed by prominent mathematicians, as lacking the rigorous foundation fit of a mathematical theory. However, the postulation of the uncertainty principle early in this century created an urgency for providing a sound mathematical framework for probablistic concepts. By this time, important contributions in that direction had been made by, among others, Poincare and Borel. But it was with Kolmogorov, in his *Foundation of the Theory of Probability* (1933), that an axiomatic foundation of the theory was presented and that the subject matter finally gained universal acceptance as a branch of mathematics.

The usefulness of this axiomization cannot be overestimated. Indeed, it provided the connection between probability as a collection of observations of natural phenomena and mathematical reasoning, thus providing a whole new set of perspectives and tools with which to view and approach related problems. Specifically, as related to the development of stochastic finite elements, the most significant aspect of mathematical probability is the association of random variables, which are the elementary ingredients of the theory, with functions defined over topological spaces. Once this association has been established, the well developed field of functional analysis could be used in analyzing and operating on these random variables. This connection with functional analysis already carries the ingredients for a unification of stochastic finite elements with deterministic finite elements. Indeed, a major part of the modern development in the theory of finite elements draws intimately from functional analysis, so that at a certain level of abstraction, both the deterministic and the stochastic finite element methods have the same theoretical foundation. Once this unification is established, the deterministic finite element method can be viewed as a special digression of the stochastic finite element method, whereby some of the functional spaces involved have a particular structure. Unlike the deterministic case, however, where functions are usually defined with respect to a Lebesgue measure, when dealing with random entities a more general concept of measure is called for. Whereas a Lebesgue measure coincides, usually, with the more intuitive notion of differential volume, probability measures are abstractions of such volumes. The practical effect of this difference is that whereas in the deterministic case a discretization of a function with respect to its natural measure induces a discretization of the physical space with respect to this same measure, and thus induces a finite element mesh, a similar discretization in the probabilistic case, with respect to the probability measure, does not carry a parallel physical consequence.

Although at first this may be seem a weak point in the development of a finite element theory, it is quickly reminded that recent efforts in deterministic finite elements have been advocating similar abstract discretizations, the p-method and various spectral methods being cases in point. In this respect, the spectral theory of the stochastic finite element method, as developed in the sequel, can be viewed as a natural extension of these developments to the case of random operators. In the next section, a coherent mathematical framework is presented which is a natural setting for the analysis of random operator equations. Next, the theory of representation of stochastic processes is expanded with special emphasis on two spectral expansions, namely the Karhunen-Loeve and the Polynomial Chaos expansions. These are then used in the following section to develop the stochastic finite element method.

2. THE MATHEMATICAL MODEL

The class of problems dealt with in this study is not of the conventional engineering kind in that it involves concepts of a rather abstract and mathematical nature. It is both necessary and instructive to introduce at this point the mathematical concepts which are used in the sequel.

The Hilbert space of functions (Oden, 1979) defined over a domain D, with values on the real line R, is denoted by H. Let (Ω, Ψ, P) denote a probability space. By that is meant that Ω is a space of elementary events, Ψ is the σ-field generated by Ω, or loosely speaking, the space consisting of the various combinations of the elements of Ω, and finally, P is the probability measure defined on Ψ. Let $\mathbf{x}$ be an element of D and θ be an element of Ω. Then, the space of functions mapping Ω onto the real line is denoted by Θ. Each map $\Omega \rightarrow R$ defines a random variable.

The inner products over H and over Θ are defined using the Lebesgue measure and the probability measure, respectively. That is, for any two elements $h_i(\mathbf{x})$ and $h_j(\mathbf{x})$ in H, their inner product ($h_i(\mathbf{x})$, $h_j(\mathbf{x})$) is defined as

$$(h_i(\mathbf{x}), h_j(\mathbf{x})) = \int_{\mathbf{D}} h_i(\mathbf{x}) h_j(\mathbf{x}) d\mathbf{x} \,. \tag{1}$$

The domain D represents the physical space over which the problem is defined. Similarly, given any two elements $\alpha(\theta)$ and $\beta(\theta)$ in Θ, their inner product is defined as

$$(\alpha(\theta), \beta(\theta)) = \int_{\Omega} \alpha(\theta)\beta(\theta) dP \tag{2}$$

where dP is a probability measure. Under very general conditions, the integral in equation (2) is equivalent to the average of the integrand with respect to the probability measure dP, so that

$$(\alpha(\theta), \beta(\theta)) = <\alpha(\theta)\beta(\theta)> \tag{3}$$

where <.> denotes the operation of mathematical expectation. Any two elements of the Hilbert spaces defined above are said to be orthogonal if their inner product

vanishes. A random process may then be described as a function defined on the product space $D \times \Omega$. Viewed from this perspective a random process can be treated as a curve in either of H or Θ.

The physical model under consideration involves a medium whose properties exhibit random spatial fluctuations and which is subjected to a random external excitation. The mathematical representation of this problem involves an operator equation

$$\Lambda(x,\theta)\,[u(x,\theta)] \;=\; f(x,\theta) \tag{4}$$

where $\Lambda(x,\theta)[.]$ is some operator defined on $H \times \Theta$. In other words, Λ is a differential operator with coefficients exhibiting random fluctuations with respect to one or more of the independent variables. The aim then is to solve for the response $u(x,\theta)$ as a function of both its arguments. With no loss of generality, Λ is assumed to be a differential operator whose random coefficients are restricted to being second order random processes. This is not a severe restriction for practical problems, since most physically measurable processes are of the second order type. Then, each one of these coefficients $a_k(x,\theta)$ can be decomposed into a purely deterministic component and a purely random component in the form

$$a_k(x,\theta\,) \;=\; \bar{a}_k(x) \;+\; \alpha_k(x,\theta) \tag{5}$$

where $\bar{a}_k(x)$ is equal to the mathematical expectation of the process $a_k(x,\theta)$, and $\alpha_k(x,\theta)$ is a zero-mean random process, having the same covariance function as the process $a_k(x,\theta)$. Equation (4) can then be written as

$$(L(x) \;+\; \Pi(x,\theta))\,[u(x,\theta)] \;=\; f(x,\theta)\;, \tag{6}$$

where $L(x)[.]$ is a deterministic differential operator and $\Pi(x,\theta)[.]$ is a differential operator whose coefficients are zero-mean random processes. Before a solution to equation (6) is sought, it is essential to clarify what is meant by such a solution.

It will prove instructive to start with the deterministic finite element method and see how the related concepts can be generalized. A finite element solution to a deterministic problem governed by a certain differential equation consists basically of computing the value of the dependent variables on a discrete mesh induced in the space spanned by the independent variables. This is probably the most widespread interpretation of a finite element solution; it has been crucial in disseminating the method as a powerful analysis and design tool into engineering practice. An alternative viewpoint which will prove to be more amenable to the required generalizations, is that a solution to a finite element problem consists in evaluating the value of the coefficients in the expansion of the solution along a certain basis in an appropriate functional space. The finite element procedure will consist in choosing a suitable basis and then computing optimal values of the coefficients with respect to this basis. From this perspective, the finite element mesh is naturally induced with specific choices of these bases. With other choices, however, the expansion coefficients do not necessarily carry an obvious physical interpretation. In the stochastic case, one of the independent variables spans the space of elementary events, which can

only be discretized with respect to a probability measure, the result lacking any intuitive appeal. In this case, the appeal of the second interpretation of a finite element solution is obvious. The problem then becomes one of identifying a suitable basis in the space $\mathbf{H} \times \Theta$ over which the solution is defined, and of determining a meaningful optimality criterion for computing the coefficients in the associated expansion. Obviously, the basis functions in this case will be random. By simulating realizations of these functions, corresponding realizations of the solution process can be obtained. Alternatively, by defining a suitable inner product over the space of random variables, various statistics or, equivalently, norms of the solution process may be evaluated.

3. REPRESENTATION OF STOCHASTIC PROCESSES

Similarly to the case of the deterministic finite element method, whereby functions are represented by a denumerable set of parameters consisting of the values of the function and its derivatives at the nodal points, the problem encountered in the stochastic case is that of representing a random process by a denumerable set of random variables, thereby discretizing the process.

In the deterministic case discretization of the domain has a physical appeal. The domain in the stochastic case does not, however, have a physical meaning that permits a sensible discretization. In this context the functional analysis foundation of the finite element method becomes useful as it can be extended to deal with random functions. Two of the most useful expansions for random processes are the Karhunen-Loeve expansion, and the Polynomial Chaos expansion. The first requires knowledge of the covariance structure of the process under consideration, while the second one is more general. The difference between these two expansions can be loosely compared to that between a modal expansion and a Fourier-type expansion of a system response. Although the former has better convergence properties, the latter is more general and does not require knowledge of the properties of the system. These two expansions are discussed next.

3.1 Karhunen-Loeve Expansion

The major conceptual difficulty from the viewpoint of the class of problems considered herein, involves the treatment of functions defined on these abstract spaces, namely random variables defined on the $\sigma-$field of random events. The most widely used method, the Monte Carlo simulation, consists of sampling these functions at randomly chosen elements of this σ-field, in a random, collocation-like, scheme. Obviously, a quite large number of points must be sampled if a good approximation is to be achieved. Alternatively, these functions could be expanded in a Fourier-type series as

$$w(\mathbf{x}, \theta) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \xi_n(\theta) f_n(\mathbf{x}) , \tag{7}$$

where $\{\xi_n(\theta)\}$ is a set of random variables to be determined, λ_n is some constant, and $\{f_n(x)\}$ is an orthonormal set of deterministic functions. This is exactly what the Karhunen-Loeve expansion achieves. The expansion was derived independently by a number of investigators (Karhunen, 1947; Loeve, 1948; Kac and Siegert, 1947).

Let $w(x,\theta)$ be a random process, function of the position vector x defined over the domain D, with θ belonging to the space of random events Ω. Let $\bar{w}(x)$ denote the expected value of $w(x,\theta)$ over all possible realizations of the process, and $C(x_1,x_2)$ denote its covariance function. By definition of the covariance function, it is bounded, symmetric and positive definite. Thus, it has the spectral decomposition (Courant and Hilbert, 1953)

$$C(x_1,x_2) = \sum_{n=1}^{\infty} \lambda_n f_n(x_1) f_n(x_2) \tag{8}$$

where λ_n and $f_n(x)$ are the eigenvalue and the normalized eigenvector of the covariance kernel, respectively. That is, they are the solution to the integral equation

$$\int_D C(x_1,x_2) f_n(x) dx_1 = \lambda_n f_n(x_2) . \tag{9}$$

Due to the symmetry and the positive definiteness of the covariance kernel (Loeve, 1977), its eigenfunctions are orthogonal and form a complete set. They have further been normalized so that the following equation holds,

$$\int_D f_n(x) f_m(x) dx = \delta_{nm} , \tag{10}$$

where δ_{nm} is the Kronecker delta. Clearly, $w(x,\theta)$ can be written as

$$w(x,\theta) = \bar{w}(x) + \alpha(x,\theta) , \tag{11}$$

where $\alpha(x,\theta)$ is a process with zero mean and covariance function $C(x_1,x_2)$. The process $\alpha(x,\theta)$ can be expanded in terms of the eigenfunctions $f_n(x)$ as

$$\alpha(x,\theta) = \sum_{n=1}^{\infty} \xi_n(\theta) \sqrt{\lambda_n} f_n(x) . \tag{12}$$

Second order properties of the random variables ξ_n can be determined by multiplying both sides of equation (12) by $\alpha(x_2, \theta)$ and taking the expectation on both sides. Specifically, it is found that

$$\begin{aligned} C(x_1,x_2) &= <\alpha(x_1,\theta)\, \alpha(x_2,\theta)> \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} <\xi_n(\theta)\, \xi_m(\theta)> \sqrt{\lambda_n \lambda_m}\, f_n(x_1)\, f_m(x_2) . \end{aligned} \tag{13}$$

Then, multiplying both sides of equation (14) by $f_k(x_2)$, integrating over the domain D, and making use of the orthogonality of the eigenfunctions, yields

$$\int_D C(x_1, x_2)\, f_k(x_2)\, dx_2 = \lambda_k\, f_k(x_1) \tag{14}$$
$$= \sum_{n=1}^{\infty} <\xi_n(\theta)\, \xi_k(\theta) > \sqrt{\lambda_n \lambda_k}\, f\ x_1) .$$

Multiplying once more by $f_l(x_1)$ and integrating over D, gives

$$\lambda_k \int_D f_k(x_1)\, f_l(x_1)\, dx_1 = \sum_{n=1}^{\infty} E<\xi_n(\theta)\, \xi_k(\theta) > \sqrt{\lambda_n \lambda_k}\, \delta_{nl} \tag{15}$$

Then, using equation (10) leads to

$$\lambda_k\, \delta_{kl} = \sqrt{\lambda_k\, \lambda_l} <\xi_k(\theta)\, \xi_l(\theta) > . \tag{16}$$

Equation (16) can be rearranged to give

$$<\xi_k(\theta)\, \xi_l(\theta) > = \delta_{kl} . \tag{17}$$

Thus, the random process $w(x, \theta)$ can be written as

$$w(x, \theta) = \bar{w}(x) + \sum_{n=1}^{\infty} \xi_n(\theta)\, \sqrt{\lambda_n}\, f_n(x) . \tag{18}$$

where,

$$<\xi_n(\theta) > = 0 \quad , \quad <\xi_n(\theta)\, \xi_m(\theta)> = \delta_{nm} , \tag{19}$$

and λ_n, $f_n(x)$ are solution to equation (9). Truncating the series in equation (18) at the M^{th} term, gives

$$w(x, \theta) = \bar{w}(x) + \sum_{n=0}^{M} \xi_n(\theta)\, \sqrt{\lambda_n}\, f_n(x) . \tag{20}$$

An explicit expression for $\xi_n(\theta)$ can be obtained by multiplying equation (12) by $f_n(x)$ and integrating over the domain D. That is,

$$\xi_n(\theta) = \frac{1}{\lambda_n} \int_D \alpha(x, \theta)\, f_n(x)\, dx . \tag{21}$$

It is well known from functional analysis that the steeper a bilinear form decays to zero as a function of one of its arguments, the more terms are needed in its spectral representation in order to reach a preset accuracy. Noting that the Fourier transform operator is a spectral representation, it may be concluded that the faster

the autocorrelation function tends to zero, the broader is the corresponding spectral density, and the greater the number of requisite terms to represent the underlying random process by the Karhunen-Loeve expansion.

For the special case of a random process possessing a rational spectrum, the integral eigenvalue problem can be replaced by an equivalent differential equation that is easier to solve (Van Trees, 1968). In the same context, it is reminded that a necessary and sufficient condition for a process to have a finite dimensional Markov realization is that its spectrum be rational (Kree and Soize, 1986). Further, note that analytical solutions for the integral equation (10) are obtainable for some quite important and practical forms of the kernel $C(\mathbf{x}_1, \mathbf{x}_2)$ (Juncosa, 1945; Slepian and Pollak, 1961; Van Trees, 1968). In the general case, however, the integral equation must be solved numerically. Various techniques are available to this end (Ghanem and Spanos, 1991).

3.2 Homogeneous Chaos

It is clear from the preceding discussion that the implementation of the Karhunen-Loeve expansion requires knowledge of the covariance function of the process being expanded. As far as the system under consideration is concerned, this implies that the expansion can be used for the random coefficients in the operator equation. However, it cannot be implemented for the solution process, since its covariance function and therefore the corresponding eigenfunctions are not known. An alternative expansion is clearly needed which circumvents this problem. Such an expansion could involve a basis of known random functions with deterministic coefficients to be found by minimizing some norm of the error resulting from a finite representation. This should be construed as similar to the Fourier series solution of deterministic differential equations, whereby the series coefficients are determined so as to satisfy some optimality criterion. To clarify this important idea further, a general functional form of the solution process is written as

$$u = h\,[\xi_i(\theta), x] \tag{22}$$

where $h[.]$ is a nonlinear functional of its arguments. In equation (22), the random processes involved have all been replaced by their corresponding Karhunen-Loeve representations. It is clear now that what is required is a nonlinear expansion of $h[.]$ in terms of the set of random variables $\xi_i(\theta)$. If the processes defining the operator are Gaussian, this set is a sampled derivative of the Wiener process (Doob, 1953). In this case, equation (22) involves functionals of the Brownian motion. This is exactly what the concept of Homogeneous Chaos provides. This concept was first introduced by Wiener (1938) and consists of an extension of Volterra's work on the generalization of Taylor series to functionals (Volterra, 1913). Wiener's contributions were the result of his investigations of nonlinear functionals of the Brownian motion. Based on Wiener's ideas, Cameron and Martin (1947) constructed an orthogonal basis for nonlinear functionals in terms of Fourier-Hermite functionals.

3.2.1 Definitions and Properties

Let $\{\xi_i(\theta)\}_{i=1}^{\infty}$ be a set of orthonormal Gaussian random variables. Consider the space $\hat{\Gamma}_p$ of all polynomials in $\{\xi_i(\theta)\}_{i=1}^{\infty}$ of degree not exceeding p. Let Γ_p represent the set of all polynomials in $\hat{\Gamma}_p$ orthogonal to $\hat{\Gamma}_{p-1}$. Finally, let $\bar{\Gamma}_p$ be the space spanned by Γ_p. Then, the subspace $\bar{\Gamma}_p$ of Θ is called the p^{th} Homogeneous Chaos, and Γ_p is called the Polynomial Chaos of order p.

Based on the above definitions, the Polynomial Chaoses of any order p consist of all orthogonal polynomials of order p involving any combination of the random variables $\{\xi_i(\theta)\}_{i=1}^{\infty}$. It is clear, then, that the number of Polynomial Chaoses of order p, which involve a specific random variable out of the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$ increases with p. This fact plays an important role in connection with the finite dimensional Polynomial Chaoses to be introduced in the sequel. Furthermore, since random variables are themselves functions, it becomes clear that Polynomial Chaoses are functions of functions and are therefore functionals.

The set of Polynomial Chaoses is a linear subspace of the space of square-integrable random variables Θ, and is a ring with respect to the functional multiplication $\Gamma_p\Gamma_l(\omega) = \Gamma_p(\omega)\Gamma_l(\omega)$. In this context, square integrability must be construed to be with respect to the probability measure defining the random variables. Denoting the Hilbert space spanned by the set $\{\xi_i(\theta)\}$ by $\Theta(\xi)$, the resulting ring is denoted by $\Phi_{\Theta(\xi)}$, and is called the ring of functions generated by $\Theta(\xi)$. Then, it can be shown that under some general conditions, the ring $\Phi_{\Theta(\xi)}$ is dense in the space Θ (Kakutani, 1961). This means that any square-integrable random function $(\Omega \rightarrow R)$ can be approximated as closely as desired by elements from $\Phi_{\Theta(\xi)}$. Thus, any element $\mu(\theta)$ from the space Θ admits the following representation,

$$\mu(\theta) = \sum_{p\geq 0} \sum_{n_1+\ldots+n_r=p} \sum_{\rho_1,\ldots,\rho_r} a^{n_1\ldots n_r}_{\rho_1\ldots\rho_r} \Gamma_p(\xi_{\rho_1}(\theta),\ldots,\xi_{\rho_r}(\theta)) , \qquad (23)$$

where $\Gamma_p(.)$ is the Polynomial Chaos of order p. The superscript n_i refers to the number of occurrences of $\xi_{\rho_i}(\theta)$ in the argument list for $\Gamma_p(.)$. Also, the double subscript provides for the possibility of repeated arguments in the argument list of the Polynomial Chaoses, thus preserving the generality of the representation given by equation (23). Briefly stated, the Polynomial Chaos appearing in equation (23) involves r distinct random variables out of the set $\{\xi_i(\theta)\}_{i=1}^{\infty}$, with the k^{th} random variable $\xi_k(\theta)$ having multiplicity n_k, and such that the total number of random variables involved is equal to the order p of the Polynomial Chaos. The Polynomial Chaoses of any order will be assumed to be symmetric with respect to their arguments. Such a symmetrization is always possible. Indeed, a symmetric polynomial can be obtained from a non-symmetric one by taking the average of the polynomial over all permutations of its arguments. The form of the coefficients appearing in equation (23) can then be simplified, resulting in the following expanded expression for the representation of random variables,

$$\mu(\theta) = a_0\,\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_{i_1}(\theta)) \qquad (24)$$

$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta))$$
$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta))$$
$$+\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} a_{i_1 i_2 i_3 i_4}\Gamma_4(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta),\xi_{i_4}(\theta))+\ldots,$$

where $\Gamma_p(.)$ are successive Polynomial Chaoses of their arguments, the expansion being convergent in the mean-square sense. The upper limits on the summations in equation (24) reflect the symmetry of the Polynomial Chaoses with respect to their arguments, as discussed above. The Polynomial Chaoses of order greater than one have mean zero. Polynomials of different order are orthogonal to each other; so are same order polynomials with different argument list. At times in the ensuing developments, it will prove notationally expedient to rewrite equation (24) in the form

$$\mu(\theta) = \sum_{j=0}^{\infty} \hat{a}_j \Psi_j[\xi(\theta)], \tag{25}$$

where there is a one-to-one correspondence between the functionals $\Psi[.]$ and $\Gamma[.]$, and also between the coefficients $\hat{a}_j$ and $a_{i_1\ldots i_r}$ appearing in equation (24). Implicit in equation (24) is the assumption that the expansion (24) is carried out in the order indicated by that equation. In other words, the contribution of polynomials of lower order is accounted for first.

Throughout the previous theoretical development, the symbol θ has been used to emphasize the random character of the quantities involved. It will be deleted in the ensuing development whenever the random nature of a certain quantity is obvious from the context.

As defined above, each Polynomial Chaos is a function of the infinite set $\{\xi_i\}$, and is therefore an infinite dimensional polynomial. In a computational setting, however, this infinite set has to be replaced by a finite one. In view of that, it seems logical to introduce the concept of a finite dimensional Polynomial Chaos. Specifically, the n-dimensional Polynomial Chaos of order p is the subset of the Polynomial Chaos of order p, as defined above, which is a function of only n of the uncorrelated random variables ξ_i. As $n \to \infty$, the Polynomial Chaos as defined previously is recovered. Obviously, the convergence properties of a representation based on the n-dimensional Polynomial Chaoses depend on n as well as on the choice of the subset $\{\xi_{\lambda_i}\}_{i=1}^{n}$ out of the infinite set. In the ensuing analysis, this choice will be based on the Karhunen-Loeve expansion of an appropriate random process. Since the finite dimensional Polynomial Chaos is a subset of the (infinite-dimensional) Polynomial Chaos, the same symbol will be used for both, with the dimension being specified. Note that for this case, the infinite upper limit on the summations in equation (24) is replaced by a number equal to the dimension of the Polynomials involved. For clarity, the two-dimensional counterpart of equation (24)

is rewritten, in a fully expanded form, as

$$\begin{aligned}\mu(\theta) = \; & a_0\,\Gamma_0 \;+\; a_1\,\Gamma_1(\xi_1) \;+\; a_2\,\Gamma_1(\xi_2) \\ & +\; a_{11}\,\Gamma_2(\xi_1,\xi_1) \;+\; a_{12}\Gamma_2(\xi_2,\xi_1) \;+\; a_{22}\Gamma_2(\xi_2,\xi_2) \\ & +\; a_{111}\,\Gamma_3(\xi_1,\xi_1,\xi_1) \;+\; a_{211}\,\Gamma_3(\xi_2,\xi_1,\xi_1) \;+\; a_{221}\,\Gamma_3(\xi_2,\xi_2,\xi_1) \\ & +\; a_{222}\,\Gamma_3(\xi_2,\xi_2,\xi_2) \;\ldots\;.\end{aligned} \tag{26}$$

In view of this last equation, it becomes clear that, except for a different indexing convention, the functionals $\Psi[.]$ and $\Gamma[.]$ are identical. In this regard, equation (26) can be recast in terms of $\Psi_j[.]$ as follows

$$\begin{aligned}\mu(\theta) = \; & \hat{a}_0\Psi_0 \;+\; \hat{a}_2\Psi_2 \;+\; \hat{a}_3\Psi_3 \;+\; \hat{a}_4\Psi_4 \;+\; \hat{a}_5\Psi_5 \\ & +\; \hat{a}_6\Psi_6 \;+\; \hat{a}_7\Psi_7 \;+\; \hat{a}_8\Psi_8 \;+\; \hat{a}_9\Psi_9 \;+\ldots\;,\end{aligned} \tag{27}$$

from which the correspondence between $\Psi[.]$ and $\Gamma[.]$ is evident. For example, the term $a_{211}\Gamma_3(\xi_2,\xi_1,\xi_1)$ of equation (26) is identified with the term $\hat{a}_7\Psi_7$ of equation (27).

3.2.2 Construction of the Polynomial Chaos

A direct approach to construct the successive Polynomial Chaoses is to start with the set of homogeneous polynomials in $\{\xi_i(\theta)\}$ and to proceed, through a sequence of orthogonalization procedures. The zeroth order polynomial is a constant and it can be chosen to be 1. That is

$$\Gamma_0 \;=\; 1\;. \tag{28}$$

The first order polynomial has to be chosen so that it is orthogonal to all zeroth order polynomials. In this context, orthogonality is understood to be with respect to the inner-product defined by equation (2). Since the set $\{\xi_i\}$ consists of zero-mean elements, the orthogonality condition implies

$$\Gamma_1(\xi_i) \;=\; \xi_i\;. \tag{29}$$

The second order Polynomial Chaos consists of second order polynomials in $\{\xi_i\}$ that are orthogonal to both constants and first order polynomials. Formally, a second order polynomial can be written as

$$\Gamma_2(\xi_{i_1},\xi_{i_2}) \;=\; a_0 \;+\; a_{i_1}\xi_{i_1} \;+\; a_{i_2}\xi_{i_2} \;+\; a_{i_1i_2}\xi_{i_1}\xi_{i_2}\;, \tag{30}$$

where the constants are so chosen as to satisfy the orthogonality conditions. The second of these requires that

$$<\Gamma_2(\xi_{i_1},\xi_{i_2})\;\xi_{i_3}> \;=\; 0\;. \tag{31}$$

This leads to the following equation

$$a_{i_1}\delta_{i_1i_3} \;+\; a_{i_2}\delta_{i_2i_3} \;=\; 0\;. \tag{32}$$

Allowing i_3 to be equal to i_1 and i_2 successively, permits the evaluation of the coefficients a_{i_1} and a_{i_2} as

$$a_{i_1} = 0 , \quad a_{i_2} = 0 . \tag{33}$$

The first orthogonality condition yields

$$a_0 + a_{i_1 i_2}\delta_{i_1 i_2} = 0 . \tag{34}$$

Equation (34) can be normalized by requiring that

$$a_{i_1 i_2} = 1 . \tag{35}$$

This leads to

$$a_0 = -\delta_{i_1 i_2} . \tag{36}$$

Thus, the second Polynomial Chaos can be expressed as

$$\Gamma_2(\xi_{i_1}, \xi_{i_2}) = \xi_{i_1}\xi_{i_2} - \delta_{i_1 i_2} . \tag{37}$$

In a similar manner, the third order Polynomial Chaos has the general form

$$\begin{aligned}\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) = & a_0 + a_{i_1}\xi_{i_1} + a_{i_2}\xi_{i_2} + a_{i_3}\xi_{i_3} + a_{i_1 i_2}\xi_{i_1}\xi_{i_2} \\ & + a_{i_1 i_3}\xi_{i_1}\xi_{i_3} + a_{i_2 i_3}\xi_{i_2}\xi_{i_3} + a_{i_1 i_2 i_3}\xi_{i_1}\xi_{i_2}\xi_{i_3} ,\end{aligned} \tag{38}$$

with conditions of being orthogonal to all constants, first order polynomials, and second order polynomials. The first of these conditions implies that

$$<\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3})> = 0 . \tag{39}$$

That is,

$$a_0 + a_{i_1 i_2}\delta_{i_1 i_2} + a_{i_1 i_3}\delta_{i_1 i_3} + a_{i_2 i_3}\delta_{i_2 i_3} = 0 . \tag{40}$$

The second condition implies that

$$<\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3})\, \xi_{i_4}> = 0 , \tag{41}$$

which leads to

$$a_{i_1}\delta_{i_3 i_4} + a_{i_2}\delta_{i_2 i_4} + a_{i_3}\delta_{i_3 i_4} + a_{i_1 i_2 i_3} <\xi_{i_1}\, \xi_{i_2}\, \xi_{i_3}\, \xi_{i_4}> . \tag{42}$$

The last orthogonality condition is equivalent to

$$<\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3})\ \xi_{i_4}\, \xi_{i_5}> = 0 , \tag{43}$$

which gives

$$\begin{aligned}a_0\, \delta_{i_4 i_5}\ a_{i_1 i_2} <\xi_{i_1}\xi_{i_2}\xi_{i_4}\xi_{i_5}> + a_{i_1 i_3} <\xi_{i_1}\xi_{i_3}\xi_{i_4}\xi_{i_5}> \\ + a_{i_2 i_3} <\xi_{i_2}\xi_{i_3}\xi_{i_4}\xi_{i_5}> = 0.\end{aligned} \tag{44}$$

The above equations can be normalized by requiring that

$$a_{i_1 i_2 i_3} = 1 . \tag{45}$$

Then equation (42) becomes

$$a_{i_1}\delta_{i_2 i_4} + a_{i_2}\delta_{i_3 i_4} + a_{i_3}\delta_{i_1 i_4} + <\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}> = 0 \tag{46}$$

Due to the Gaussian property of the set $\{\xi_i\}$, the following equation holds

$$<\xi_{i_1}\xi_{i_2}\xi_{i_3}\xi_{i_4}> = \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} . \tag{47}$$

Substituting for the expectations in equations (46) and (44) yields

$$\begin{aligned} & a_{i_1}\delta_{i_2 i_4} + a_{i_2}\delta_{i_3 i_4} + a_{i_3}\delta_{i_1 i_4} \\ & + \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} = 0 , \end{aligned} \tag{48}$$

and

$$\begin{aligned} a_0\,\delta_{i_4 i_5} & + a_{i_1 i_2}\left[\delta_{i_1 i_2}\delta_{i_4 i_5} + \delta_{i_1 i_4}\delta_{i_2 i_5} + \delta_{i_1 i_5}\delta_{i_2 i_4}\right] \\ & + a_{i_1 i_3}\left[\delta_{i_1 i_3}\delta_{i_4 i_5} + \delta_{i_1 i_4}\delta_{i_3 i_5} + \delta_{i_1 i_5}\delta_{i_3 i_4}\right] \\ & + a_{i_2 i_3}\left[\delta_{i_2 i_3}\delta_{i_4 i_5} + \delta_{i_2 i_4}\delta_{i_3 i_5} + \delta_{i_2 i_5}\delta_{i_3 i_4}\right] = 0. \end{aligned} \tag{49}$$

Substituting for a_0 from equation (40), equation (49) can be rewritten as

$$\begin{aligned} a_{i_1 i_2}\left[\delta_{i_1 i_4}\delta_{i_2 i_5} + \delta_{i_1 i_5}\delta_{i_2 i_4}\right] + a_{i_1 i_3}\left[\delta_{i_1 i_4}\delta_{i_3 i_5} + \delta_{i_2 i_5}\delta_{i_3 i_4}\right] \\ + a_{i_2 i_3}\left[\delta_{i_2 i_4}\delta_{i_3 i_5} + \delta_{i_2 i_5}\delta_{i_3 i_4}\right] = 0 . \end{aligned} \tag{50}$$

From equation (50), the coefficients $a_{i_1 i_2}$, $a_{i_1 i_3}$, and $a_{i_2 i_3}$ can be evaluated as

$$\begin{aligned} a_{i_1 i_2} &= 0 \\ a_{i_1 i_3} &= 0 \\ a_{i_2 i_3} &= 0 . \end{aligned} \tag{51}$$

Using equation (49) again, it is found that

$$a_0 = 0 . \tag{52}$$

Equation (48) can be rewritten as

$$\delta_{i_1 i_4}(a_{i_2} + \delta_{i_2 i_3}) + \delta_{i_1 i_4}(a_{i_1} + \delta_{i_2 i_3}) + \delta_{i_1 i_4}(a_{i_1} + \delta_{i_2 i_3}) = 0, \tag{53}$$

from which the coefficients a_{i_1}, a_{i_2}, and a_{i_3} are found to be,

$$\begin{aligned} a_{i_1} &= -\delta_{i_2 i_3} \\ a_{i_2} &= -\delta_{i_1 i_3} \\ a_{i_3} &= -\delta_{i_1 i_2} . \end{aligned} \tag{54}$$

The third order Polynomial Chaos can then be written as

$$\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) = \xi_{i_1}\,\xi_{i_2}\,\xi_{i_3} - \xi_{i_1}\,\delta_{i_2 i_3} - \xi_{i_2}\,\delta_{i_1 i_3} - \xi_{i_3}\,\delta_{i_1 i_2} . \tag{55}$$

After laborious algebraic manipulations, the fourth order Polynomial Chaos can be expressed as

$$\begin{aligned} \Gamma_4(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \xi_{i_4}) &= \xi_{i_1}\,\xi_{i_2}\,\xi_{i_3}\,\xi_{i_4} \\ &- \xi_{i_1}\,\xi_{i_2}\,\delta_{i_3 i_4} - \xi_{i_1}\,\xi_{i_3}\,\delta_{i_2 i_4} - \xi_{i_1}\,\xi_{i_4}\,\delta_{i_2 i_3} \\ &- \xi_{i_2}\,\xi_{i_3}\,\delta_{i_1 i_4} - \xi_{i_2}\,\xi_{i_4}\,\delta_{i_1 i_3} - \xi_{i_3}\,\xi_{i_4}\,\delta_{i_1 i_2} \\ &+ \delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} . \end{aligned} \tag{56}$$

It is readily seen that, in general, the n^{th} order Polynomial Chaos can be written as

$$\Gamma_p(\xi_{i_1}, \ldots, \xi_{i_n}) = \begin{cases} \displaystyle\sum_{\substack{r=0 \\ r \text{ even}}}^{n} (-1)^r \sum_{\pi(i_1,\ldots,i_n)} \prod_{k=1}^{r} \xi_{i_k} < \prod_{l=r+1}^{n} \xi_{i_l} > \\ \qquad\qquad n \text{ even} \\ \displaystyle\sum_{\substack{r=0 \\ r \text{ even}}}^{n} (-1)^{r-1} \sum_{\pi(i_1,\ldots,i_n)} \prod_{k=1}^{r} \xi_{i_k} < \prod_{l=r+1}^{n} \xi_{i_l} > \\ \qquad\qquad n \text{ odd} \end{cases} \tag{57}$$

where $\pi(.)$ denotes a permutation of its arguments, and the summation is over all such permutations such that the sets $\{\xi_{i_1}, \ldots, \xi_{i_r}\}$ is modified by the permutation.

Note that the Polynomial Chaoses as obtained in equations (28), (29), (37), (55) and (56) are orthogonal with respect to the Gaussian probability measure, which makes them identical with the corresponding multidimensional Hermite polynomials (Grad, 1949). These polynomials have been used extensively in relation to problems in turbulence theory (Imamura et.al, 1965a-b). This equivalence is implied by the orthogonality of the Polynomial Chaoses with respect to the inner product defined by equation (2) where dP is the Gaussian measure $e^{-\frac{1}{2}\xi^T\xi}\,d\xi$, where ξ denotes the

vector of n random variables $\{\xi_{i_k}\}_{k=1}^{n}$. This measure is exactly the weighing function with respect to which the Hermite polynomials are orthogonal in the L_2 sense (Oden, 1979). This fact suggests another method for constructing the Polynomial Chaoses, namely from the generating function of the Hermite polynomials. Specifically, the Polynomial Chaos of order n can be obtained as

$$\Gamma_n(\xi_{i_1},\ldots,\xi_{i_n}) \;=\; (-1)^n \,\frac{\partial^n}{\partial\xi_{i_1}\ldots\partial\xi_{i_n}} e^{-\frac{1}{2}\xi^T\xi} \tag{58}$$

The first two terms in equation (25) represent the Gaussian component of the function $\mu(\theta)$. Therefore, for a Gaussian process, this expansion reduces to a single summation, the coefficients a_{i_1} being the coefficients in the Karhunen-Loeve expansion of the process. Note that equation (25) is a convergent series representation for the functional operator $h[.]$ appearing in equation (23). For a given non-Gaussian process defined by its probability distribution function, a representation in the form given by equation (25) can be obtained by projecting the process on the successive Homogeneous Chaoses. This can be achieved by using the inner product defined by equation (2) to determine the requisite coefficients. This concept has been successfully applied in devising efficient variance reduction techniques to be coupled with the Monte Carlo simulation method (Chorin, 1971; Maltz and Hitzl, 1979).

4. PROJECTION ON THE HOMOGENEOUS CHAOS

In this section the Karhunen-Loeve expansion and the Polynomial Chaos expansion presented earlier are implemented into a stochastic finite element method which features a number of similarities with the deterministic finite element method. Specifically, the geometric interpretation of the finite element method as a projection in function space is preserved.

Equation (6) constitute the starting point. Assuming that

$$\Pi(\mathbf{x},\omega)[.] \;=\; \alpha(\mathbf{x},\theta)\; \mathrm{R}(\mathbf{x})[.]\,, \tag{59}$$

and expanding $\alpha(\mathbf{x},\theta)$ in a Karhunen-Loeve series gives

$$\left(\; \mathrm{L}(\mathbf{x}) \;+\; \sum_{n=1}^{M} \xi_n\; a_n(\mathbf{x})\; \mathrm{R}(\mathbf{x}) \right)\; [u(\mathbf{x},\theta)] \;=\; f(\mathbf{x},\theta)\,. \tag{60}$$

Assuming, without loss of generality, that $u(\mathbf{x},\theta)$ is a second order process, it lends itself to a Karhunen-Loeve expansion of the form

$$u(\mathbf{x},\theta) \;=\; \sum_{j=1}^{L} e_j\; \chi_j(\theta)\; b_j(\mathbf{x})\,, \tag{61}$$

where

$$\int_{\mathrm{D}} C_{uu}\;(\mathbf{x}_1\,,\;\mathbf{x}_2)\; b_j(\mathbf{x}_2)\; d\mathbf{x}_2 \;=\; e_j\; b_j(\mathbf{x}_1)\,, \tag{62}$$

and

$$\chi_j(\theta) = \frac{1}{e_j} \int_D u(x,\theta)\, b_j(x)\, dx \,. \tag{63}$$

Obviously, the covariance function $C_{uu}(x_1, x_2)$ of the response process is not known at this stage. Thus, e_j and $b_j(x)$ are also not known. Further, $u(x,\theta)$, not being a Gaussian process, the set $\chi_j(\theta)$ is not a Gaussian vector. Therefore, equation (61) is of little use in its present form. Relying on the discussion concerning the Homogeneous Chaos, the second order random variables $\chi_j(\theta)$ can be represented by the mean-square convergent expansion

$$\begin{aligned}\chi_j(\theta) &= a_{i_0}^{(j)}\,\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}^{(j)}\,\Gamma_1(\xi_{i_1}) \\ &+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}^{(j)}\,\Gamma_2(\xi_{i_1},\xi_{i_2}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}^{(j)}\,\Gamma_3(\xi_{i_1},\xi_{i_2},\xi_{i_3}) \\ &+ \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} a_{i_1 i_2 i_3 i_4}^{(j)}\,\Gamma_4(\xi_{i_1},\,\xi_{i_2},\,\xi_{i_3},\,\xi_{i_4}) + \ldots,\end{aligned} \tag{64}$$

where $a_{i_1,\ldots,i_p}^{(j)}$ are deterministic constants independent of θ and $\Gamma_p(\,\xi_{i_1},\,\ldots,\,\xi_{i_p}\,)$ is the p^{th} order Homogeneous Chaos. Equation (64) is truncated after the P^{th} polynomial and is rewritten for convenience, as discussed in equation (27), in the following form,

$$\chi_j(\theta) = \sum_{i=0}^{P} x_i^{(j)}\,\Psi_i[\{\xi_r\}]\,, \tag{65}$$

where $x_i^{(j)}$ and $\Psi_i[\{\xi_r\}]$ are identical to $a_{i_1,\ldots,i_p}^{(j)}$ and $\Gamma_p(\xi_{i_1},\ldots\xi_{i_p})$, respectively. In equation (65), P denotes the total number of Polynomial Chaoses used in the expansion, excluding the *zeroth* order term. Given the number M of terms used in the Karhunen-Loeve expansion, and the order p of Homogeneous Chaos used, P may be determined by the equation

$$P = 1 + \sum_{s=1}^{p} \frac{1}{s!} \prod_{r=0}^{s-1} (M + r)\,. \tag{66}$$

Substituting equation (65) for $\chi_j(\theta)$, equation (61) becomes

$$u(x,\theta) = \sum_{j=1}^{L}\sum_{i=0}^{P} x_i^{(j)}\,\Psi_i[\{\xi_r\}]\,c_j(x)\,, \tag{67}$$

where

$$c_j(x) = e_j\, b_j(x)\,. \tag{68}$$

Changing the order of summation in equation (67) gives

$$u(x,\theta) = \sum_{i=0}^{P} \Psi_i[\{\xi_r\}] \sum_{j=1}^{L} x_i^{(j)}\,c_j(x)$$

$$= \sum_{i=0}^{P} \Psi_i[\{\xi_r\}]\, d_i(\mathbf{x}) \,, \tag{69}$$

where,

$$d_i(\mathbf{x}) = \sum_{k=1}^{L} x_i^{(j)}\, c_j(\mathbf{x}) \,. \tag{70}$$

Substituting equation (69) for $u(\mathbf{x},\theta)$, equation (60) becomes

$$\left(\mathbf{L}(\mathbf{x}) + \sum_{n=1}^{M} \xi_n\, a_n(\mathbf{x})\, \mathbf{R}(\mathbf{x}) \right) \left[\sum_{j=0}^{P} \Psi_j[\{\xi_r\}]\, d_j(\mathbf{x}) \right] = f(\mathbf{x}) \,, \tag{71}$$

where reference to the parameter θ was eliminated for notational simplicity. The response $u(\mathbf{x},\theta)$ can be completely determined once the functions $d_i(\mathbf{x})$ are known. In terms of the eigenfunctions $b_j(\mathbf{x})$ of the covariance function of $u(\mathbf{x},\theta)$, $d_i(\mathbf{x})$ can be expressed as

$$\begin{aligned} d_i(\mathbf{x}) &= \sum_{j=1}^{L} x_i^{(j)}\, e_j\, b_j(\mathbf{x}) \\ &= \sum_{j=1}^{L} y_i^{(j)}\, b_j(\mathbf{x}) \,. \end{aligned} \tag{72}$$

Equation (71) may be written in an alternative form

$$\sum_{j=0}^{P} \Psi_j[\{\xi_j\}]\, \mathbf{L}(\mathbf{x})\, [d_j(\mathbf{x})] + \sum_{j=0}^{P} \sum_{i=1}^{M} \xi_i\, \Psi_j[\{\xi_r\}]\, \mathbf{R}(\mathbf{x})\, [d_j(\mathbf{x})] = f(\mathbf{x}). \tag{73}$$

This form of the equation shows that $d_j(\mathbf{x})$ belongs to the intersection of the domains of $\mathbf{R}(\mathbf{x})[.]$ and $\mathbf{L}(\mathbf{x})[.]$. Then, following the standard deterministic finite element method, the function $d_j(\mathbf{x})$ may be expanded in an appropriate function space as

$$d_j(\mathbf{x}) = \sum_{k=1}^{N} d_{kj}\, g_k(\mathbf{x}) \,. \tag{74}$$

Then, equation (73) becomes

$$\begin{aligned} &\sum_{j=0}^{P} \sum_{k=1}^{N} d_{kj}\, \Psi_j[\{\xi_r\}]\, \mathbf{L}(\mathbf{x})\, [g_k(\mathbf{x})] \\ &\qquad + \sum_{j=0}^{P} \sum_{i=1}^{M} \xi_i(\theta)\, \Psi_j[\{\xi_r\}] \sum_{k=1}^{N} d_{kj}\, \mathbf{R}(\mathbf{x})\, [g_k(\mathbf{x})] = f(\mathbf{x}) \,. \end{aligned} \tag{75}$$

Equation (75) may be rearranged to give

$$\begin{aligned} \sum_{j=0}^{P} \sum_{k=1}^{M} d_{kj}\, \Big[&\Psi_j[\{\xi_r\}]\, \mathbf{L}(\mathbf{x})\, [g_k(\mathbf{x})] \\ &+ \sum_{i=1}^{M} \xi_i(\theta)\, \Psi_j[\{\xi_r\}]\, \mathbf{R}(\mathbf{x})\, [g_k(\mathbf{x})] \Big] = f(\mathbf{x}) \,. \end{aligned} \tag{76}$$

Multiplying both sides of equation (76) by $g_l(x)$ and integrating throughout yields

$$\sum_{j=0}^{P} \sum_{k=1}^{M} d_{kj} \left[\Psi_j[\{\xi_r\}] \int_D L(x)\ [g_k(x)]\ g_l(x)\ dx + \sum_{i=1}^{M} \xi_i\ \Psi_j[\{\xi_r\}] \int_D R(x)\ [g_k(x)]\ g_l(x)\ dx \right] = \int_D f(x)\ g_l(x)\ dx\ ,\ l = 1, \ldots, N\ . \tag{77}$$

Setting

$$L_{kl} = \int_D L(x)\ [g_k(x)]\ g_l(x)\ dx \tag{78}$$

$$R_{ikl} = \int_D R(x)\ [g_k(x)]\ g_l(x)\ a_i(x)\ dx \tag{79}$$

$$f_l = \int_D f(x)\ g_l(x)\ dx\ , \tag{80}$$

equation (77) becomes

$$\sum_{j=0}^{P} \sum_{k=1}^{N} \left[\Psi_j[\{\xi_r\}]\ L_{kl} + \sum_{i=1}^{M} \xi_i(\theta)\ \Psi_j[\{\xi_r\}]\ R_{ikl} \right] d_{kj} = f_l\ ,\ l = 1, \ldots, N\ . \tag{81}$$

Note that the index j spans the number of Polynomial Chaoses used, while the index k spans the number of basis vectors used in C^m. Multiplying equation (81) by $\Psi_m[\{\xi_r\}]$, averaging throughout and noting that

$$<\Psi_j[\{\xi_r\}]\ \Psi_m[\{\xi_r\}]> = \delta_{jm} <\Psi_m^2[\{\xi_r\}]>\ , \tag{82}$$

one can derive

$$\sum_{k=1}^{N} <\Psi_m^2[\{\xi_r\}]> L_{kl} d_{km} + \sum_{j=0}^{P} \sum_{k=1}^{N} d_{kj} \sum_{i=1}^{M} <\xi_i(\theta) \Psi_j[\{\xi_r\}] \Psi_m[\{\xi_r\}]> R_{ikl} = <f_l\ \Psi_m[\{\xi_r\}]>,\quad l = 1, \ldots, N\ ,\ m = 1, \ldots, P\ . \tag{83}$$

Introducing

$$c_{ijm} \equiv <\xi_i\ \Psi_j[\{\xi_r\}]\ \Psi_m[\{\xi_r\}]>\ , \tag{84}$$

and assuming, without loss of generality, that the Polynomial Chaoses have been normalized, equation (83) becomes

$$\sum_{k=1}^{N} L_{kl}\ d_{km} + \sum_{j=0}^{P} \sum_{k=1}^{N} d_{kj} \sum_{i=1}^{M} R_{ikl}\ c_{ijm} = <f_l\ \Psi_m[\{\xi_r\}]>\ , \quad l = 1, \ldots, N\ ,\ m = 1, \ldots, P\ . \tag{85}$$

For a large number of index combinations the coefficients c_{ijm} are identically zero. Equation (84) was implemented using the symbolic manipulation program MACSYMA (1986). Forming equation (83) for all P values of m, produces a set of $N \times P$ algebraic equations of the form

$$[\,\mathbf{G} + \mathbf{R}\,]\,\mathbf{d} = \mathbf{h}\,, \tag{86}$$

where $\mathbf{G}$ and $\mathbf{R}$ are block matrices of dimension $N \times P$. Their mj^{th} blocks are N-dimensional square matrices given by the equations

$$\mathbf{G}_{mj} = \delta_{mj}\,\mathbf{L}\,, \tag{87}$$

and

$$\mathbf{R}_{mj} = \sum_{i=1}^{M} c_{ijm}\,\mathbf{R}_i\,. \tag{88}$$

In equations (87) and (88), $\mathbf{L}$ and $\mathbf{R}_i$ denote N-dimensional square matrices whose kl^{th} element is given by equations (78) and (79), respectively. In equation (86), $\mathbf{h}$ signifies the $N \times M$ vector whose m^{th} block is given by the equation

$$\mathbf{h}_m = <\mathrm{f}\,\Psi_m[\{\xi_r\}]>\,. \tag{89}$$

The N-dimensional vectors $\mathbf{d}_m$ can be obtained as the subvectors of the solution to the deterministic algebraic problem given by equation (86). Once these coefficients are obtained, back substituting into equation (69) yields an expression of the response process in terms of the Polynomial Chaoses of the form

$$\mathbf{u} = \sum_{j=0}^{P} \mathbf{d}_j\,\Psi_j[\{\xi_r\}]\,. \tag{90}$$

Based on equation (90), realizations of the random response vector can be computed from realizations of the random variables $\{\xi_r\}$. Also, statistical moments of the random response vector can be evaluated using the inner product defined in equation (2).

5. Numerical Examples

The preceding development of the stochastic finite element method was applied to a number of problems from engineering mechanics. The first step in the solution of any of these problems was the solution of the eigenvalue problem associated with the Karhunen-Loeve expansion. Following that, the coefficients in the Polynomial Chaos expansion for the solution process were computed. Finally, various statistics, as well as the probability distribution of the solution process were numerically evaluated. Figure (1) shows a thin plate whose modulus of elasticity is assumed to be a two-dimensional random process. The plate is analyzed using the stochastic finite

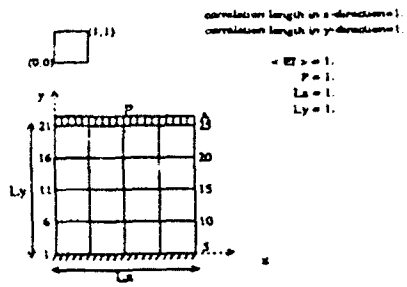

Figure 1: Plate with Random Rigidity; Exponential Covariance Model.

element formulation described above. Figure (2) compares some of the coefficients in equation (90) for various levels of approximation; note the excellent convergence. Figure (3) shows the variation of the standard deviation of the response against the standard deviation of the material property again for various levels of approximation. Finally, figure (4) shows the probability distribution of the response variable at the free corner of the plate.

6. Conclusions

A method for the solution of differential equations with random processes as coefficients was discussed. The method relies on viewing the random aspect of the problem as an added dimension, and on treating random variables and processes as functions defined over that dimension. In this manner, a formulation for the stochastic finite element method was derived which could be construed as a natural extension of the deterministic finite element method. Finite element representation along the random dimension was achieved via two spectral expansions. One of them was used to represent the coefficients of the differential equation which model the random material properties, the other was used to represent the random solution process. The new concepts were implemented using a number of computational models for simple engineering systems. The convergence of the discussed approximations was demonstrated numerically. Probability distribution functions of the response variables were obtained.

The present formulation can be viewed as a definite step towards a unification of various finite element techniques. Indeed it consists of generalizing the concepts of finite element approximation to abstract spaces, of which the usual euclidian space is a special case. The deterministic case can then be regarded as a digression

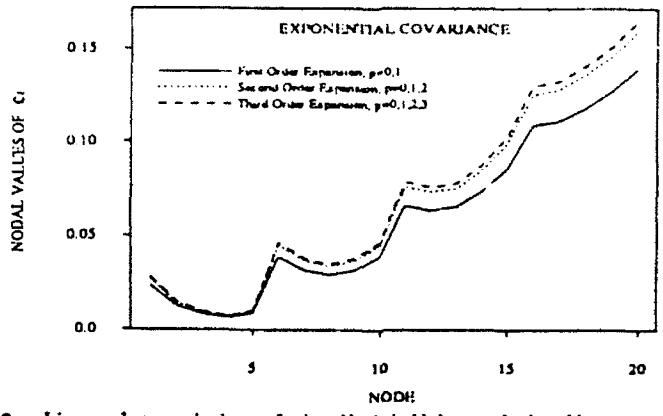

Figure 2: Linear Interpolation of the Nodal Values of the Vector $\mathbf{c}_i$ of Equation (5.73) for the Rectangular Plate Stretching Problem, $i = 2$; Longitudinal Displacement Representation; 2 Terms in K-L Expansion, $M = 2$; $P = 3, 6, 10$.

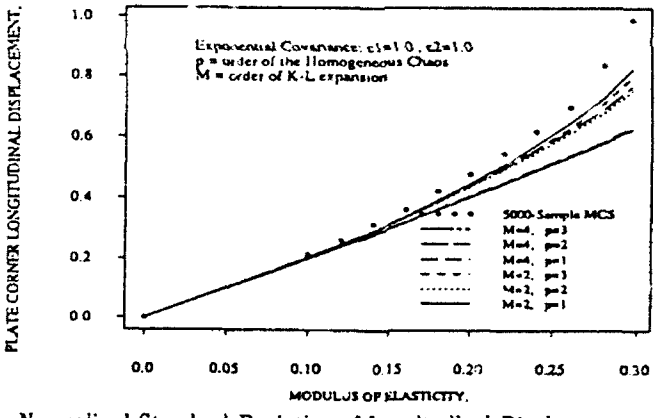

Figure 3: Normalized Standard Deviation of Longitudinal Displacement at Corner A of the Rectangular Plate, versus Standard Deviation of the Modulus of Elasticity; $\sigma_{max} = 0.433$; Exponential Covariance; Polynomial Chaos Solution

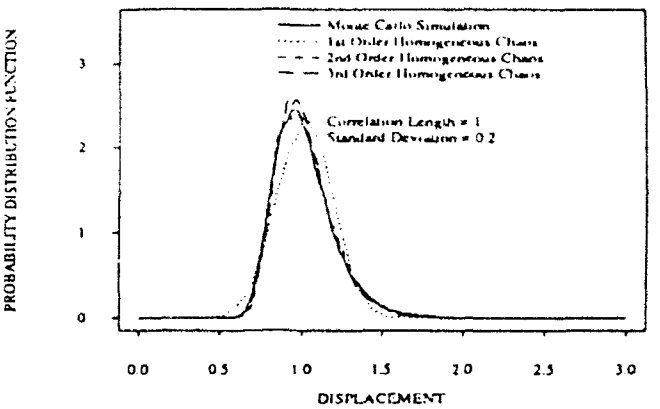

Figure 4: Longitudinal Displacement at the Free End of the Rectangular Plate; Probability Density Function Using 30,000-Sample MSC, and Using Third Order Homogeneous Chaos; Four Terms in the K-L Expansion; Exponential Covariance.

of this formalism to the particular instance when the space of elementary events consists of a single element, and where the probability density function induced on the associated σ-algebra is the uniform distribution.

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3)

Pseudo - Simulation Method for Stochastic Problems

B.A. Zeldin[1], P.D. Spanos[2]

Abstract

A new numerical method for problems of stochastic mechanics and other areas involving a small number of random parameters is presented. It is analogous to the Monte-Carlo simulation method and quite more efficient. As an example, the eigenvalue problem of a clamped-clamped beam with random rigidity is considered.

Key words: Monte-Carlo method, Finite Element method, Finite Difference method, partition, sample space, eigenvalue problem, mean value, standard deviation.

Introduction

The Monte-Carlo simulation method has been widely used in the area of stochastic mechanics and others fields, primarily because of its versatility. This method treats a stochastic problem as an ensemble of deterministic ones. After solving a number of deterministic problems, statistical analysis is performed to deduce the necessary parameters characterizing the random nature of the solution. Often this method is the only option available to solve complicated stochastic problems. However, indiscriminate use of the method can not be advocated due to its considerable computational cost. In this paper a new numerical method for stochastic problems is presented. In essence, it is a Monte-Carlo simulation method utilizing a limited number of random variables.

Formulation

Consider a problem governed by the equation

$$L(\xi)\, u = f(\xi)\,. \tag{1}$$

where $\xi = (\xi_1, \xi_2, \ldots \xi_M)$ is a random vector, $L(\xi)$ is a mathematical operator describing the performance of this system which depends on ξ. Further, $f(\xi)$ depends on the same parameter set and describes the load. The number M is assumed to be small, and $\xi_1, \xi_2, \ldots \xi_M$ are statistically independent random variables.

Solving equation (1) is equivalent to finding some function $u = u(\xi)$ which satisfies this equation. That is, for every realization $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \ldots \bar{\xi}_M)$ of the random vector ξ there exists a deterministic function $u(\bar{\xi})$ which satisfies the equation (1).

Consider the space R^M of $\xi_1 \times \xi_2 \times \ldots \xi_M$ as the sample space Ω. This space can be divided into N subdomains Ω_i, i=1,...N having the shape of M-dimensional disjoint rectangles with prescribed probability mass. Next introduce the set of functions φ_i, i=1,...N such that

$$\varphi_i(\xi) = \begin{cases} 1, & \text{if} \quad \xi \in \Omega_i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Clearly, since $\varphi_i(\xi)$ and $\varphi_j(\xi)$ have disjoint support, unless $i = j$,

$$\int_{\Omega} q(\xi)\, \varphi_i(\xi)\, \varphi_j(\xi)\, p_\xi(\xi)\, d\xi = 0 \qquad \text{if} \quad i \neq j, \tag{3}$$

where $p_\xi(\xi)$ is the probability density function of ξ, and $q(\xi)$ is an arbitrary random variable. This means that the set $\{\varphi_i\}_{i=1}^{N}$ is orthogonal with arbitrary weight basis for the class of random variables which are constant for every Ω_i.

1. Research Assistant, Civ.Eng., Rice University, P.O.Box 1892, Houston, Texas 77251.
2. L.B.Ryon Chair in Eng., Rice University, P.O.Box 1892, Houston, Texas 77251.

The solution of equation (1) can be approximated by the series

$$u(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi) , \tag{4}$$

where $u(\xi)$ is assumed to be an adequately smooth function of ξ. Next, the scalar product of two random variables a and b can be introduced by the equation

$$E[ab] = \langle a, b \rangle . \tag{5}$$

Then, the solution given by equation (4) can be constructed as a projection of the exact solution into the space $Sp\{\varphi_i\}_{i=1}^{N}$. Thus, the following sequence of equations can be written

$$\langle L(\xi) \sum_{i=1}^{N} c_i \varphi_i(\xi) , \varphi_j(\xi) \rangle = \langle f(\xi) , \varphi_j(\xi) \rangle \quad , \tag{6}$$

which, because of equations (3) and (5), leads to

$$E[L(\xi) c_i \varphi_i(\xi)] = E[f(\xi) \varphi_i(\xi)] . \tag{7}$$

This sequence of deterministic equations can be solved to determine the solutions c_i. Upon determining c_i, the statistical properties of the solution can be estimated by relying on equation (4). Specifically,

$$E[u(\xi)] = \sum_{i=1}^{N} c_i p_i \quad \text{and} \quad E[u^2(\xi)] = \sum_{i=1}^{N} c_i^2 p_i \quad , \tag{8}$$

where $p_i = E[\varphi_i(\xi)] = E[\varphi_i^2(\xi)]$.

If $L(\xi) = \sum_k L_k P_k(\xi)$, where $P_k(\xi)$ is some known function of ξ and L_i is linear, equation (6) leads to the simple expression

$$\sum_k L_k c_i E[P_k(\xi) \varphi_i(\xi)] = E[f(\xi) \varphi_i(\xi)] . \tag{9}$$

Finite Element Method - Pseudo Monte-Carlo Perspective

Examining the proposed method one may view it as a Finite Element method for random media. That is, it involves approximations of the random variables in the finite dimensional subspaces defined by some finite partition of the sample space.

From another perspective, examining equation (7) one can deduce that this equation is equivalent to the following

$$\int_{\Omega_i} [L(\xi) c_i - f(\xi)] p_\xi(\xi) \, d\xi = 0. \tag{10}$$

If the Lebesgue integral involved in equation (10) can be interpreted in the Riemann sense and all quantities in the above expression are adequately smooth, the mean-value theorem states that there exists some $\hat{\xi} \in \Omega_i$ such that

$$L(\hat{\xi}) c_i = f(\hat{\xi}) . \tag{11}$$

This equation shows that c_i represent just a solution of equation (1) for the realization $\hat{\xi}$ of the random vector ξ. Then, the sequence of the equations (7) can be interpreted as a sequence of Monte-Carlo simulations. In fact, this sequence of "pseudo-simulations" is optimal in the sense that every element of this sequence represents a certain region of the sample space and can be interpreted as the only outcome with given probability.

For the implementation of this method, first the entire sample space is divided into some assembly

of subsets, the mesh of events, with given probability mass. Then, certain elements are chosen from every subset, and the sequence of equations (1) is treated as a deterministic one. It is further assumed that these elements of the sample space represent a realization of the random vector ξ. Then, statistical analysis is conducted assuming that the derived deterministic solutions are the only possible outcomes with given probability. It is clear that this algorithm relates to the Monte-Carlo method. Thus, it should work at least as well as the Monte-Carlo method. Further, it exhibits the appealing feature of choosing the points from the subsets in an optimal way, namely in the sense of the projection represented by the equation (6).

Example: Eigenvalue Problem for a Beam

The proposed method was applied to the eigenvalue problem of a clamped-clamped beam of unit length; its rigidity is a truncated normal random process with mean equal to 1 and autocorrelation function

$$R_{EI}(x_1, x_2) = \sigma^2 exp(-\frac{(x_2 - x_1)^2}{c}).$$

The case with $\sigma = 0.3$ and c=0.5 was taken. The corresponding equation is

$$(EI(x)u'')'' = \lambda u, \tag{12}$$

with boundary conditions $u(0) = u(1) = u'(0) = u'(1) = 0$..

This kind of problem is quite difficult either for an analytical or for a numerical treatment. Only a few papers are available on this topic. A description of pertinent analytical methods was presented by Bruce(1968). In the papers of Hasselman and Hart (1972), and Grigoriu (1991) some numerical examples of solution of stochastic eigenvalue problems can be found. However, these algorithms can be applied in the case of small randomness only and can be computationally costly.

In implementing the proposed method, first the discretization procedure is applied to obtain a finite dimensional problem. For this purpose the Finite Difference method, see also Spanos and Zeldin (1992), is used. This leads to the equation

$$Au = \lambda u, \tag{13}$$

where A is a matrix with random variables as elements and u is a random vector. Subsequently the Karhunen-Loeve expansion, see also Ghanem and Spanos (1988,1991), can be applied to represent the matrix A in the form

$$A = A_0 + \xi_1 A_1 + \ldots \xi_M A_M + \ldots, \tag{14}$$

where $\xi_1, \xi_2, \ldots \xi_M, \ldots$ are statistically independent random variables. The series in equation (14) can be truncated beyond order M, and the random vector $\xi = (\xi_1, \xi_2, \ldots \xi_M)$ can be introduced. Next, the random domain is divided into a set of rectangles

$$\xi \in \{\xi | \xi_i^j \le \xi_i \le \tilde{\xi}_i^j, \quad i=1,\ldots M; \quad j=1,\ldots N\}$$

of prescribed probability mass. Then, the basis $\{\varphi_i\}_{i=1}^N$ can be constructed to conform with equation (2). Next, u and λ can be taken in the form

$$u(\xi) = \sum_{i=1}^{N} v_i \varphi_i(\xi), \qquad \lambda(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi) \quad . \tag{15}$$

Substituting equations (14) and (15) into equation (13), multiplying it by $\varphi_j(\xi)$, and taking the mathematical expectation of the result yields

$$(A_0 + \xi_1^k A_1 + \ldots \xi_M^k A_M) v_k = c_k v_k \quad , \qquad i=1,\ldots N, \tag{16}$$

where $\xi_i^k = \frac{1}{p_k} \int_\Omega \xi_i \varphi_k(\xi) p_\xi(\xi)\, d\xi = \frac{1}{p_k} \int_{\Omega_k} \xi_i p_\xi(\xi)\, d\xi$ and $p_k = \int_\Omega \varphi_k(\xi) p_\xi(\xi)\, d\xi = \int_{\Omega_k} p_\xi(\xi)\, d\xi$.

Finally, statistical analysis can be performed in conjunction with equation (8) to estimate the moments

of the first two eigenvalues and eigenvectors.

Numerical Results - Concluding Remarks

The theoretical values of the first two eigenvalues for the deterministic case, when the beam rigidity is set equal to the mean rigidity of the problem under consideration, are $\lambda_1^{det} = 22.37$ and $\lambda_2^{det} = 61.67$. The corresponding deterministic finite difference approximation with 41 node points gives $\lambda_1^{findif} = 22.32$ and $\lambda_2^{findif} = 61.33$. It was found that despite the considerable variability in the rigidity of the beam, the variability in the eigenvectors is negligible. However, the variability in the eigenvalues is essential and for $\sigma = 0.3$, as in this example, is of the order of 10%, see Figure 1. The influence of different ξ_i was studied. It was found that for this problem the third term in equation (16) is almost negligible. The results in terms of convergence of this method for different order of partition of axis ξ_1 and ξ_2, that is for different numbers of pseudo-simulations, are plotted in Figure 1(a) for the standard deviation of λ_1, and in Figure 1(b) for the standard deviation of λ_2. It is seen that this method gives quite good approximations even when the number of pseudo-simulations is very small, whereas the Monte-Carlo method yields reliable results only if the number of the simulations used is large. The mean of the first and second eigenvalues was found to be 21.98 and 60.40, respectively. Thus, the mean values slightly decline to smaller values compared to the deterministic case.

It appears that the discussed method can be applied for treating a wide class of problem dealing with random variables and stochastic processes. Thus, further research might be warranted.

Acknowledgment

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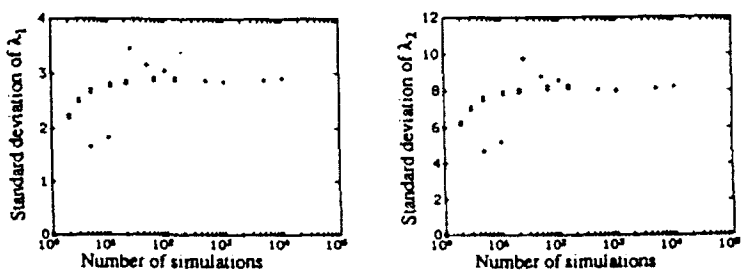

(a) (b)

Figure 1

+ Monte - Carlo simulation * Proposed pseudo-simulation method

4)

Stochastic Mixed Finite Difference Method

P.D.Spanos[1], B.A.Zeldin[2]

Abstract

Some aspects of numerical solutions of stochastic mechanics problems are considered. The Finite Difference method for discretization of stochastic continuous media problems is discussed. The Neumann expansion and perturbation methods used for solving the associated system of algebraic equations are analyzed. Their dependence on the mesh size of the discretization is investigated. It is shown that a mixed formulation using both strain and stress as independent variables improves the performance of these methods and reduces their dependence on the mesh size.

Key words: random variable, random process, Finite Element method, Finite Difference method, beam equation, pertubation method, Neumann expansion, convergence.

Introduction

Recently, considerable attention has been given to the solution of engineering problems with randomness in the spatial domain. Clearly, the spacial dependence hinders solving these problems exactly. Thus, approximate methods of solution have been pursued. The main idea has been to implement the well known and widely used Finite Element (FEM) or Finite Difference (FDM) methods. A number of papers have appeared recently on this topic addressing linear or nonlinear and static or dynamic problems; see for example Ghanem and Spanos(1988,1991), Liu W.K.,et.al.(1986), Vanmarke and Grigoriu(1983), Takada (1990), Yamazaki, et.al.(1986).

Usually such an analysis involves two steps. The first step is to introduce some discretization of the continuous medium problem, which leads to a system of stochastic algebraic equations; the second step addresses the solution of this system.

In this paper some new aspects of solving continuous media problems with randomness in the spatial domain using FDM are considered.

Discretization Techniques

Several different techniques for random problem discretization have been developed. Typically they use the following random field representation

$$f(x) = \sum_i c_i v_i(x), \tag{1}$$

where $f(x)$ is a random field, c_i are random variables, and $v_i(x)$ are basis functions built upon some appropriate partition of the problem.Ghanem and Spanos(1988,1991) used this form for stochastic field representation with basis functions from Kurhunen-Loeve expansion. After such a discretization in the random domain, the Finite Element procedure was applied. Alternatively Takada (1990) and Deodatis, etc.(1991) used the so called Weighted Integral method which is straightforward application of FEM to the random problems.

Interestingly, stochastic FDM has not received much attention. In the deterministic case often FDM has some advantage over FEM due to simplicity in formulation and analysis. The same can be said concerning stochastic problems. A similar formulation was introduced by Vanmarke and Grigoriu (1983) for the simple case of a statically determinant shear beam.

A Finite Difference approximation for a differential operator is next discussed. Let $u(x)$ be a stochastic process. Then, the mean-square derivatives of $u(x)$ in some point x_i can be approximated by using the central difference formulae

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1. L.B.Ryon Chair in Engng., Rice Univ., P.O.Box 1892, Houston, Texas 77251.
2. Graduate Student Civ. Engng., Rice Univ., P.O.Box 1892, Houston, Texas 77251.

$$\frac{d}{dx}u(x_i) \approx \frac{u_{i+1}-u_i}{2h}; \qquad \frac{d^2}{dx^2}u(x_i) \approx \frac{u_{i+1}-2u_i+u_{i-1}}{h^2}, \tag{2}$$

where u_i is a random variable equal to $u(x_i)$ and h is the mesh size of the partition involving N+1 nodes.

Using these equations the beam operator $Lu(x) = (EI(x)u''(x))$ can be approximated by the equation

$$Lu(x_i) \approx \frac{1}{h^4}(EI_{i-1}u_{i-2} + (-2(EI_{i-1}+EI_i))u_{i-1} + (EI_{i-1}+4EI_i+EI_{i+1})u_i$$
$$-2(EI_i+EI_{i+1})u_{i+1} + EI_{i+1}u_{i+2}) \quad , \tag{3}$$

where x_i are nodal points, i=0,1,...N.

Appending to equation (3) the corresponding approximations of proper boundary conditions, a system of linear stochastic algebraic equations can be formed. Specifically,

$$Au = f \quad , \tag{4}$$

where A is a matrix with random variables as elements, u is a random vector of the solution, and f is a vector of load which in general is random.

Solution of the System of Algebraic Equation

The solution of the problem described by equation (4) is crucial in computational stochastic mechanics. Several methods have been developed for this purpose. One of them is the Monte-Carlo simulation method which has been widely used. This method was incorporated in FEM by Shinozuka and his associates (1972), who developed an algorithm for the stochastic field simulation. A recent implementation of this method includes the Neumann expansion of matrices introduced by Yamazaki,et.al.(1986). However, this method may demand large computational resources. Another quite common method relies on a perturbation expansion which provides sufficiently good results when the randomness is not very large (Takada(1991), Liu, et.al.(1990)). Otherwise this method can give erroneous results.

Alternatively, the Neumann expansion method can be utilized. In fact that is some sort of generalization of the perturbation method. It was used by several investigations such as in Ghanem and Spanos (1988, 1991), Adomian and Malkian (1979) and Yamazaki, et.al. (1988). Let the matrix A in the equation (4) be expressed as $A = A_0 + \tilde{A}$ with $A_0 = \langle A \rangle$ and $\tilde{A} = A - A_0$. Substituting this expression into equation (4) and multiplying it by A_0^{-1} gives

$$(I + A_0^{-1}\tilde{A})u = A_0^{-1}f \quad . \tag{5}$$

Further, it is possible to introduce the following equality

$$u = \sum_{k=0}^{\infty}(-1)^k(A_0^{-1}\tilde{A})^k A_0^{-1}f = A_0^{-1}f - A_0^{-1}\tilde{A}A_0^{-1}f + (A_0^{-1}\tilde{A})^2A_0^{-1}f - \ldots \tag{6}$$

Equation (6) has only a formal meaning as it is assumed that this series converge to $(I + A_0^{-1}\tilde{A})^{-1}A_0^{-1}f$. Examination of this reveals that the convergence problem is not a trivial one. One of the sufficient conditions for this is the following almost surely inequality

$$\left\| A_0^{-1}\tilde{A} \right\| \le q < 1. \tag{7}$$

Note that the condition given by equation(7) is too restrictive and difficult for analysis. Adomian and Malkian (1979) considered linear differential operator $L = \sum a_i(\frac{d}{dt})$ with randomness in the last coefficient a_0. Convergence of these series was proved and the error of truncation was estimated in that case. However, in a number of papers this method was applied even when other coefficients a_i, i=1,...N were random; see Ghanem and Spanos (1988,1991), Adomian (,1979), Yamazaki and etc.(1988). It is not difficult to note that for

continuous problem operator d/dt is not bounded and for discrete case it behaves as 1/h, where h is a mesh size. Thus, for the beam problem the norm $\|A_0^{-1}\tilde{A}\|$ may grow as $1/h^4$.

One can show that the perturbation method represents the Neumann expansion truncated to the first 2 terms. Then, it can be concluded that whenever the mesh of the discretization is sufficiently small, both the perturbation and Neumann expansion methods of solution of a system of algebraic equations may behave poorly due to the unboundedness of the differential operator.

Mixed Method

It is possible to overcome the convergence obstacle using the concept of Mixed FDM. The applicability of this concept has been demonstrated for the solution of deterministic problems with non-smooth coefficients. In fact this method allows the elimination of non-smooth coefficients in the left hand side of the differential equations describing the problem. This method can be helpful and for the stochastic case where it is necessary to find an algorithm allowing the use of the Neumann expansion procedure. Indeed, for the beam problem the new variables $v_1(x) = u(x)$ and $v_2(x) = EIu''(x)$ can be introduced. Then, the beam equation can be written in the form

$$v' - Bv = q. \tag{8}$$

where $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $B = \begin{bmatrix} 0 & \frac{1}{EI(x)} \\ 0 & 0 \end{bmatrix}$ and $q = \begin{pmatrix} 0 \\ q(x) \end{pmatrix}$.

As one can see from equation (10), the random coefficient of this equation is associated with a term which does not involve differentiation. Therefore, the dependence of $\|A_0^{-1}\tilde{A}\|$ on the mesh size can be avoided.

To show the applicability of the method discussed, consider a clamped-clamped beam subjected to deterministic force q=1 with the bending rigidity being a normal random process as shown in Figure 1. The bending of this beam can be described by equation (8) with boundary conditions $v_1(0) = v_1(1) = v'_1(0) = v'_1(1) = 0$. To solve this problem, FDM can be applied; it results in an equation similar to equation (4). Consider a case where

$$k(x) = \frac{1}{EI(x)}, \quad \langle k(x) \rangle = 1 \quad \text{and} \quad \langle k(x_1)k(x_2) \rangle = \sigma^2 \exp\left(-\frac{(x_2 - x_1)^2}{c}\right)$$

where $\sigma = 0.3$ and $c = 1.0$. Meshes with 10 and 20 nodes were chosen. The second statistical moments of displacement and bending moment were calculated based on the expressions

$$\mathrm{Cov}(v) = \langle vv^t \rangle = \sum_{k+l=\,\mathrm{odd}>0}^{M_2} \langle (A_0^{-1}\tilde{A})^k A_0^{-l} f f^t A_0^{-l} ((A_0^{-1}\tilde{A})^l)^t \rangle - \langle u \rangle \langle u \rangle^t, \tag{9}$$

where $\langle u \rangle = \sum_{k=0}^{M_1} (-1)^k \langle (A_0^{-1}\tilde{A})^k A_0^{-1} f \rangle$.

The performance of these statistical moments with respect to the order of the truncation was investigated. The results obtained for variance are plotted in the Figure 2. It is seen that the mixed formulation improves the convergence of Neumann expansion and removes its dependence on the mesh size. These results are not affected by the partition and even the first order perturbation method is acceptable. This is especially true with regards to the displacement variability even if the coefficient of variation of the beam rigidity is large. However, the bending moment variance performs poorly compared to the displacement one and the second term of the expansion (9) should be used. It is seen that 30%, approximately, of the variability of the beam rigidity induces 10%, approximately, for the coefficient of variation response. That coincides with other results (Ghanem and Spanos (1988,1991)).

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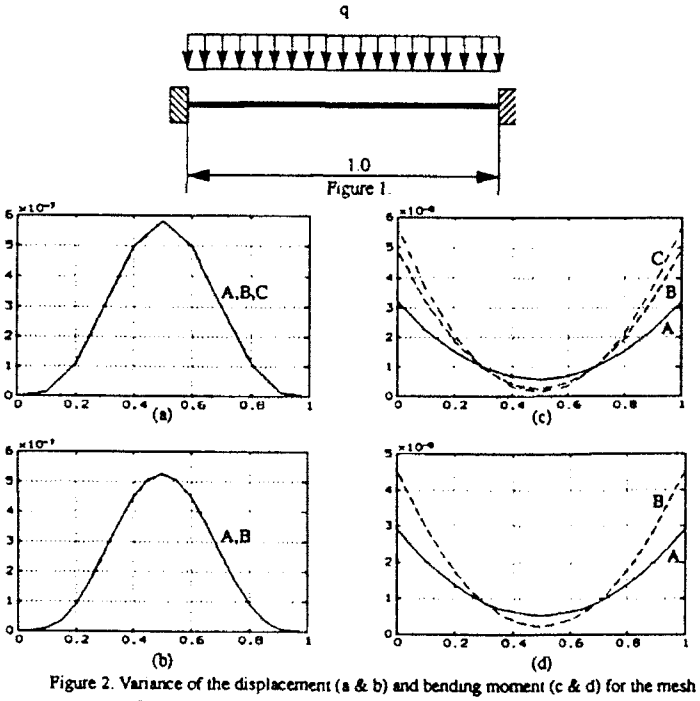

Figure 2. Variance of the displacement (a & b) and bending moment (c & d) for the mesh 10 (a & c) and 20 (b & d) nodes

(A)——— $M_2 = 2$ (B) ---- $M_2 = 4$ (C) ..... $M_2 = 6$

5)

INDIRECT SAMPLING METHOD FOR STOCHASTIC MECHANICS PROBLEMS

P.D.Spanos[1] B.A.Zeldin[2]

INTRODUCTION

The Monte-Carlo simulation method has been widely used in the field of stochastic mechanics and others fields, primarily because of its versatility. Often it is the only option available to solve complex problems. However, indiscriminate use of the method can not be advocated due to its considerable computational cost. In fact, several variance reduction techniques have been developed in this regard. They involve importance sampling, stratified sampling, and others [1].

A numerical method for problems of stochastic mechanics and other areas representing the solution by a small number of random parameters is presented. In essence, it is a stratified sampling method, but more efficient. Alternatively, this new method can be viewed as a Galerkin approximation in the sample space. Several examples are considered involving the use of the Loeve-Karhunen expansion for stochastic fields approximation [2,3]. The examples deal with the evaluation of natural frequencies and seismic response of beams with random rigidity.

FORMULATION

Solution Representation

Consider a problem with random parameters governed by the equation

$$L(\xi)\, u = f(\xi)\,, \tag{1}$$

where $\xi = (\xi_1, \xi_2, \ldots \xi_M)$ is a random vector, and $L(\xi)$ is a mathematical operator describing the performance of the system. Further, $f(\xi)$ describes the load, and $\xi_1, \xi_2, \ldots \xi_M$ are statistically independent random variables.

Solving equation (1) is equivalent to finding a function $u = u(\xi)$ such that for every realization $\xi^* = (\xi^*_1, \xi^*_2, \ldots \xi^*_M)$ of the random vector ξ there exists a deterministic function $u(\xi^*)$ which satisfies equation (1). Consider the space R^M of $\xi_1 \times \xi_2 \times \ldots \xi_M$ as the sample space Ω. This space can be divided into N subdomains or strata $\{\Omega_i, i = 1, N\}$ having the shape of M-dimensional disjoint rectangles with prescribed probability mass as shown in Figure 1 for M=2. Next, introduce the set of functions or spline basis $\{\varphi_i, i = 1, \ldots N\}$ such that

$$\varphi_i(\xi) = \begin{cases} 1, & \text{if} \quad \xi \in \Omega_i \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

Clearly, $\varphi_i(\xi)$ and $\varphi_j(\xi)$ have disjoint supports. That is,

$$\int_{\Omega} q(\xi)\, \varphi_i(\xi)\, \varphi_j(\xi)\, p_\xi(\xi)\, d\xi = 0 \qquad \text{if} \quad i \neq j, \tag{3}$$

where $p_\xi(\xi)$ is the probability density function of ξ, $q(\xi)$ is an arbitrary random variable. Thus, the set $\{\varphi_i, i = 1, \ldots N\}$ is an orthogonal basis for the class of random variables which are constant for every Ω_i. Any random variable can be approximated adequately by the use of these basis functions, provided the partition of Ω is fine. The solution of equation (1) can be represented as a linear combination of the functions $\{\varphi_i, i = 1, \ldots N\}$. That is,

$$u(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi)\,, \tag{4}$$

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1. L.B. Ryon Chair in Engineering, Rice University, P.O.Box 1892, Houston, Tx 77251
2. Research Assistant, Civil Engineering, Rice University, P.O.Box 1892, Houston, Tx 77251

where the coefficients c_i are to be determined.

Then, the solution given by equation (4) can be construed as a projection of the exact solution into the space spanned by $\{\varphi_i, i = 1, \ldots N\}$. Expressing the solution in the form of the equation (4), the induced error in the equation (1) can be made orthogonal to the space spanned by $\{\varphi_i, i = 1, \ldots N\}$. That is, using the operator of mathematical expectation, $< >$, it can be found

$$\langle L(\xi) \sum_{i=1}^{N} c_i \varphi_i((\xi)\varphi_j(\xi)) \rangle = \langle f((\xi)\varphi_j(\xi)) \rangle \quad , \qquad j = 1, \ldots, N \quad , \tag{5}$$

which, because of equation (3), leads to

$$\langle L(\xi) c_i \varphi_i^2(\xi) \rangle = \langle f(\xi)\varphi_i(\xi) \rangle \quad . \tag{6}$$

This sequence of deterministic equations can be solved to find the coefficients c_i. Upon deriving c_i, the statistical properties of the solution can be estimated by relying on equation (4). Specifically,

$$\langle u(\xi) \rangle = \sum_{i=1}^{N} c_i p_i \quad \text{and} \quad \langle u^2(\xi) \rangle = \sum_{i=1}^{N} c_i^2 p_i \quad , \tag{7}$$

where

$$p_i = \langle \varphi_i(\xi) \rangle = \langle \varphi_i^2(\xi) \rangle \quad . \tag{8}$$

Similarly, the distribution function of the solution can be found using the equation

$$P_u(v) = Pr(u < v) = \sum_{i=1}^{N} \chi_v(c_i) p_i, \tag{9}$$

where

$$\chi_v(x) = \begin{cases} 1 & \text{if } x \le v \\ 0 & \text{if } x > v \end{cases} . \tag{10}$$

Solution Interpretation

The proposed method may by viewed as a Galerkin-type procedure for random media. Several authors have explored the idea of using projection procedures in conjuction with random variables. In references [2,3,4,5,6] this procedure has been applied for stochastic mechanics problems with randomness in the spatial domain. Due to the correlation between the solution and the random parameters describing the properties of the structures, this class of problems is especially difficult to solve. In this regard, the stochastic field has been discretized by the use of the Loeve-Karhunen expansion in references [2,3,4] or of the midpoint method in reference [5]. In this manner the problem is first characterized by a finite set of random variables. Then, the solution can be derived by a Galerkin projection into finite dimensional spaces spanned by orthogonal chaos polynomials as in references [2,3,4], or just linear functions as in reference [5]. However, these bases can yield a large order system of equations which must be solved to determine the solution.

Another possible basis for the representation shown in equation (4) is given by equation (2); see also reference [6]. The concept of using spline type approximation has been discussed widely in the area of computational mechanics in connection with the finite element method. From this perspective, the system of functions $\{\varphi_i\}_{i=1}^{N}$ represents the simplest spline of piecewise-constant functions. Then, the proposed method involves approximation of the random variables in the finite dimensional subspace of splines defined by some partition of the sample space. Additional advantages of this representation relate to equation (3) since each term c_i in the expansion (4) can be found independently. Therefore, every term c_i in equation (4) can be readily determined.

From another perspective, the use of piece-wise constant functions makes this method a generalized sampling procedure. Indeed, examining equation (6) one can deduce that this equation is equivalent to the following

$$\int_{\Omega_i} [L(\xi)\, c_i - f(\xi)]\, p_{\xi}(\xi)\, d\xi = 0. \tag{11}$$

If the Lebesgue integral involved in equation (11) can be interpreted in the Riemann sense and all pertinent quantities are adequately smooth, the mean-value theorem states that there exists some $\xi^* \in \Omega_i$ such that

$$L(\xi^*)\, c_i = f(\xi^*). \tag{12}$$

This equation shows that c_i represents just a solution of equation (1) for the realization ξ^* of the random vector ξ. Then, the sequence of equations (12) can be interpreted as a sequence of samplings. In fact, this "indirect sampling" is optimal in the sense that every element of this sequence represents a certain region of the sample space and can be interpreted as the only outcome with a given probability. Moreover, as $\xi^* \in \Omega_i$, the proposed method can be viewed as analog to stratified sampling [1]. But unlike the stratified sampling method the point inside every stratum is computed to make some error of the approximation of the given numerical problem (1) orthogonal to the chosen space and minimal for a given stratification.

To show this properly regression analysis can be applied [7]. Any random variable $\Theta(\xi)$ which is a function of ξ on the sample space can be estimated by $\hat{\Theta}$ using the set of random variables $\{\varphi_i, i = 1, \ldots N\}$ defined by the equation (2), where $\hat{\Theta}$ is an arbitrary function of $\{\varphi_i, i = 1, \ldots N\}$ rather than ξ. This estimate provides a minimal variance for the difference $\Theta - \hat{\Theta}$. That is,

$$\langle (\Theta - \hat{\Theta})^2 \rangle \text{ is minimal.} \tag{13}$$

It can be shown, that as $\{\varphi_i, i = 1, \ldots N\}$ are indicator functions of disjoint sets, the estimate $\hat{\Theta}$ can be found using linear regression analysis and the solution can be expressed as

$$\hat{\Theta} = \sum_{i=0}^{\infty} \theta_i \varphi_i(\xi) \quad . \tag{14}$$

where $\theta_i = \langle \Theta | \Omega_i \rangle = \langle \Theta \varphi_i \rangle$, and $< | >$ denotes conditional expectation.

Let $u(\xi)$ be the exact solution of the equation (1), and let $\tilde{u}$ be an approximation of this solution. Define the error of such an approximation by

$$\varepsilon = L(\xi)\, \tilde{u} - L(\xi)\, u = L(\xi)\, \tilde{u} - f(\xi). \tag{15}$$

Then, the estimate $\hat{\varepsilon}$ of ε can be derived using equation (14). If the approximation $\tilde{u}$ is taken from a system of equations (5), then $\hat{\varepsilon} = 0$. Thus, the proposed method ensures that the error defined by the equation (15) has a zero mean square estimate from the indicator functions of chosen stratification.

Related perspective can be generated using some algebra concepts [7]. The vector ξ defines a sigma-algebra G in the sample space, and the stratification shown in Figure 1 defines a more coarse pure atomic σ-algebra $G_1 \subset G$ with indicator functions $\{\varphi_i, i = 1, \ldots N\}$. Then, the above regression analysis applied to the error ε leads to

$$\langle \varepsilon \rangle = 0 \quad , \quad \hat{\varepsilon} = \langle \varepsilon | G_1 \rangle = 0. \tag{16}$$

In other words, this method yields the minimal error defined by equation (15) with respect to the coarse σ-algebra of given stratification.

EXAMPLES

Preliminary Remarks

The proposed method is applied for the analysis of the dynamic behavior of a beam of unit length. The beam problem can be described by the equation

$$(EI(x)\,u''(x,t))'' = q(x,t)\,, \tag{17}$$

where u is the beam deflection, and q denotes the distributed force acting on the beam which in general is taken as a stochastic process. The symbol $EI(x)$ denotes the beam bending rigidity which is assumed to be a normal homogeneous stochastic process with mean equal to 1 and autocorrelation function

$$R_{EI}(x_1, x_2) = \sigma^2 exp\left(-\frac{(x_2 - x_1)^2}{c}\right). \tag{18}$$

where σ and c are constants. Thus, randomness is manifested in this problem through the operator and the load.

In implementing the proposed method, first the approximation of the stochastic field EI(x) through a finite set of random variables is derived. For this purpose, the Loeve-Karhunen expansion is deemed especially effective. It is an optimal, in the mean square sense, representation of the field over the set of random variables. Subsequent application of the finite difference scheme [8] or of any alternative discretization scheme leads to the system of linear algebraic equations

$$(A_0 + \xi_1 A_1 + \ldots \xi_M A_M)\,u = f\ . \tag{19}$$

where $A_1, A_2, \ldots A_M$ are matrices the dimension of which depends on the number of nodes used for the discretization, u is a vector representing the solution at the nodal points, and f corresponds to the force.

Next, specific numerical examples of application of the proposed method are presented; the numerical values $\sigma = 0.3$ and $c = 0.5$ are used.

Beam Eigenvalue Problem.

The eigenvalue problem of a clamped-clamped beam is considered first. Then, the force in equation (17) takes the form $q(x) = \lambda u(x)$. This kind of problem is quite difficult either for an analytical or for a numerical treatment. Only a few articles are available on this topic. A description of pertinent analytical methods was presented by Boyce [9]. In the papers of Goodwin and Boyce[10], Hasselman and Hart [11] some numerical examples of solution of stochastic eigenvalue problems can be found. Note, that these algorithms can only be applied in the case of small randomness and can be computationally costly.

In implementing the proposed method, the random domain is divided into a set of rectangles $\{\xi |\ \xi_i^j \le \xi_i \le \tilde{\xi}_i^j,\ \ i=1,\ldots M;\ \ j=1,\ldots N\}$ of prescribed equal probability mass. Then, the basis $\{\varphi_i,\ i = 1,\ldots N\}$ can be constructed to conform with equation (2). Next, u and λ can be expressed in the form

$$u(\xi) = \sum_{i=1}^{N} v_i \varphi_i(\xi)\,, \quad \text{and} \quad \lambda(\xi) = \sum_{i=1}^{N} c_i \varphi_i(\xi)\ . \tag{20}$$

Substituting equations (20) into equation (19), multiplying it by $\varphi_j(\xi)$, and taking the mathematical expectation of the result yields

$$(A_0 + \xi_{1,i} A_1 + \ldots \xi_{M,i} A_M)\,v_i = c_i v_i\ , \qquad i=1,\ldots N, \tag{21}$$

where

$$\xi_{k,i} = \frac{1}{p_i}\int_{\Omega} \xi_k \varphi_i(\xi)\, p_\xi(\xi)\, d\xi = \frac{1}{p_i}\int_{\Omega_i} \xi_k p_\xi(\xi)\, d\xi\ , \quad \text{and} \quad p_i = \int_{\Omega_i} p_\xi(\xi)\, d\xi\ . \tag{22}$$

Finally, statistical analysis can be performed in conjunction with equation (7) to estimate analyti-

cally the moments of the first two eigenvalues and eigenvectors.

The theoretical values of the first two eigenvalues for the deterministic case, when the beam rigidity is set equal to the mean rigidity of the problem under consideration, are $\lambda_1^{det} = 22.37$ and $\lambda_2^{det} = 61.67$. The corresponding deterministic finite difference approximation with 41 node points gives $\lambda_1^{findif} = 22.32$ and $\lambda_2^{findif} = 61.33$. It was found that despite the considerable variability in the rigidity of the beam, the variability in the eigenvectors is negligible. However, the variability in the eigenvalues is essential and it is of the order of 10%; see Figure 2. The influence of different number M of used random variables ξ_i has been studied. It was found that for this problem the contribution of the terms beyond ξ_2 in equation (19) is negligible. The results in terms of convergence of this method for different order of partition of axes ξ_1 and ξ_2, that is for different values of number N or indirect samplings, are plotted in Figure 2(a) for the standard deviation of λ_1, and in Figure 2(b) for the standard deviation of λ_2. It is seen that the proposed method yields quite good approximations even when the value of number N is quite small. However, the Monte-Carlo method, that is when the parameters were sampled arbitrarily, yields reliable results only if the number of the used simulations is large.

Beam Response to Deterministic Load.

The second problem involves continuous systems with random parameters exposed to deterministic excitation. Specifically, the dynamic response of a cantilever beam to earthquake-type base excitation is considered. In this case the force term in equation (17) can be expressed as

$$q(x,t) = a_g(t) - \ddot{u}(x,t) - \alpha \dot{u}(x,t) \ , \tag{23}$$

where α is a coefficient of damping, u represents the displacement of the beam relative to the base, and $a_g(t)$ is taken as the time history of the ground acceleration produced by the North-South component of El Centro earthquake recorded on station No 117 and reported in the reference [12]. It is shown on Figure 3 for comparisons with the beam response. Further, it is assumed that the beam has unit mass per length. The discretization of the beam by a finite difference scheme built upon 20 nodes in the spatial domain in conjuction with the Loeve-Karhunen expansion of the bending rigidity is used. Then, the solution is taken in the form of equation (20) where in this case $\underline{y}_i = \underline{y}_i(t)$ are deterministic vector-functions. Substituting this expression into the resulting equation, multiplying it by $\varphi_j(t)$, and averaging, an uncoupled system of deterministic ordinary differential equations is derived. Each equation of this system is solved numerically using the central difference scheme. Finally, the mean value and the standard deviation of the free end displacement are determined by relying on equation (7).

The time history of the free end displacement of the cantilever beam having the mean characteristic for the stiffness and damping is plotted in Figure 4(a). Further results of the calculations are shown in Figure 4(b,c) for different value of number N of indirect samplings. In Figure 4, N_1, N_2, and N_α denote the number of strata in the domain of ξ_1, ξ_2, and α, respectively. The case with $\alpha = 0.4$ which corresponds to damping of approximately 6% of critical for the first mode of the system with deterministic rigidity equal to the mean of the corresponding stochastic problem is considered. Also the case where α is a random variable statistically independent from $\underline{\xi}$ and uniformly distributed between 0.1 and 0.7 is examined. The computations show that 3 strata in the domain of α can adequately represent the dependence of the solution on the damping variability. Also, the calculations reveal that only the first two components of the vector $\underline{\xi}$ influence significantly the beam response. It can be seen that the dynamic response of the beam to the deterministic excitation is strongly affected by the rigidity variability.

Beam Response to Stochastic Load.

The third problem is described again by equations (17) and (23) but it involves stochastic base excitation. Specifically, $a_g(t)$ is taken as a stationary random process. The proposed indirect sampling method can be readily applied for treating this problem. The solution is expressed in the form of equation (20). In this case $\underline{y}_i = \underline{y}_i(t)$ is not a deterministic vector-function, but a stochastic vector-process. That is, $\underline{y}_i(t)$ is the response of a deterministic system to random excitation. A number of techniques

exist for the solution of this problem. In particular, a spectral approach can be applied provided that $a_g(t)$ is a second order stochastic process. In the latter case the second-order characteristics of the solution can be determined from the formulae

$$E[u] = \sum_{i=1}^{N} E[v_i(t)]p_i \quad \text{and} \quad R_{uu}(t_1, t_2) = \sum_{i=1}^{N} R_{v_i}(t_1, t_2)p_i. \tag{24}$$

Then, this approach can be viewed as semi-analytical method analogous to the directional sampling. At first, the vector ξ is simulated, and then, the solution for the given simulation is calculated using known analytical techniques with subsequent application of some averaging as in equation (24).

Again, two cases for α are considered. First, α is a deterministic coefficient, and second α is uniformly distributed random variable; it induces for each mode of the discrete model of the beam damping 6%, and 2% to 10% of critical, respectively. The power spectral density of the disp...ement of the free end is calculated using the proposed method for $a_g(t)$ being a white noise process of unit two sided spectral density. The data for different numbers of indirect samplings are plotted in Figure 5 together with the corresponding solution for the response of a deterministic system with rigidity equal to the mean rigidity of the stochastic system. Figure 5 shows that the randomness of the system has a significant effect on the system response variability and reduces the peaks of the response power spectral density. The calculations show that only two first components of the vector ξ influence the first two moments of the solution significantly. Further, the effect of the damping variability can be captured using only two strata in its domain.

Concluding Remarks

A Galerkin-type numerical method for stochastic mechanics problems has been presented. Specifically, it has been proposed to use a *Galerkin projection into the space* of simple random variables. This space can be spanned by the piece-wise constant spline functions with a chosen partition of the sample space. Further, it has been shown that the proposed method can be construed a generalized sampling; it is proposed to call it indirect sampling. Indeed, it has been shown that this method is closed to the stratified sampling method and it is optimal in the sense of equation (16). That is, the approximation of the problem from the space of simple random variables produces an error with mean and conditional expectation, given the sigma-algebra induced by this partition, equal to zero. It has also been shown that this error has zero estimate from the set of indicator functions of the given stratification. Some stochastic mechanics problems have been studied utilizing the proposed method in conjuction with the Loeve-Karhunen expansion which is a versatile tool for the approximation of a stochastic field by a finite set of random variables. These examples have demonstrated that the proposed method can be applied for treating a broad class of stochastic mechanics problems.

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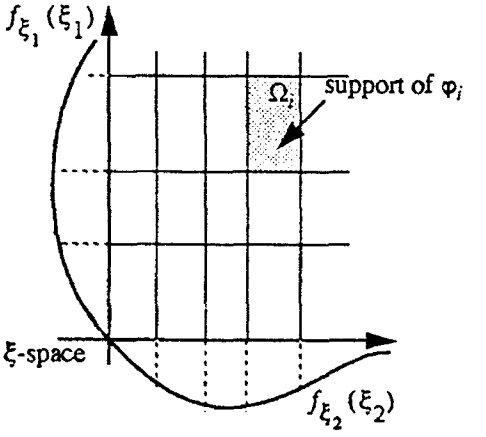


Figure 1. Stratified sample space

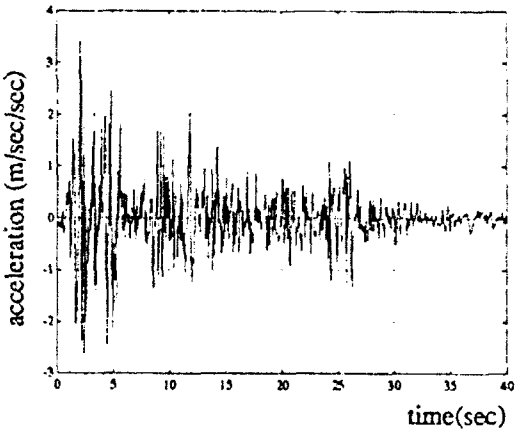
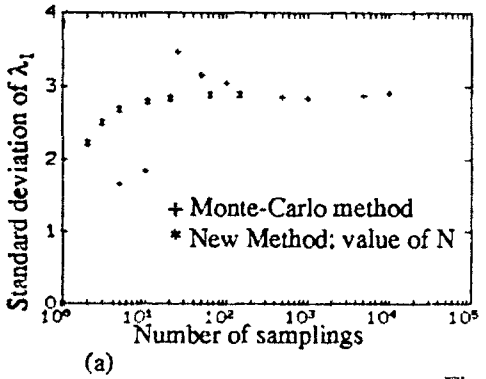
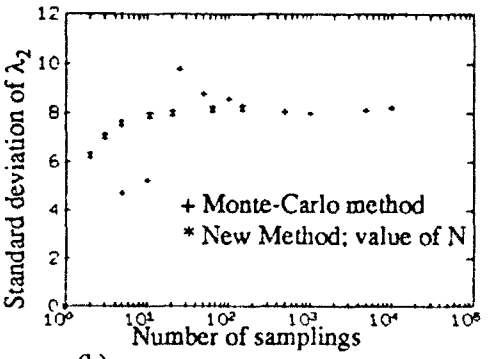


Figure 3. Earthquake-type excitation



(a)



(b)

Figure 2.

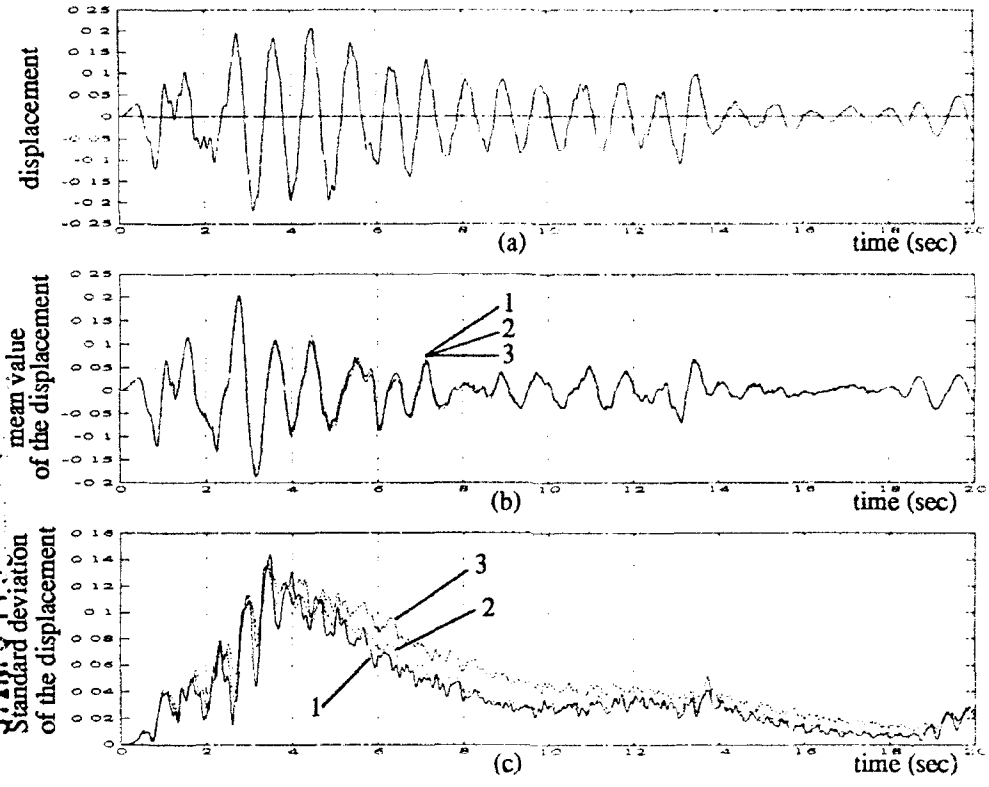


Figure 4. Base excited vibration of the top of the beam: (a) deterministic system with mean characteristics; (b) mean value; (c) standard deviation, stochastic system

1) —— N = 63 (N_1=9, N_2=7) 2) - - - - N = 7 (N_1=7)

3) - - - -N = 45 considering random damping (N_1=5, N_2=3, N_α=3)

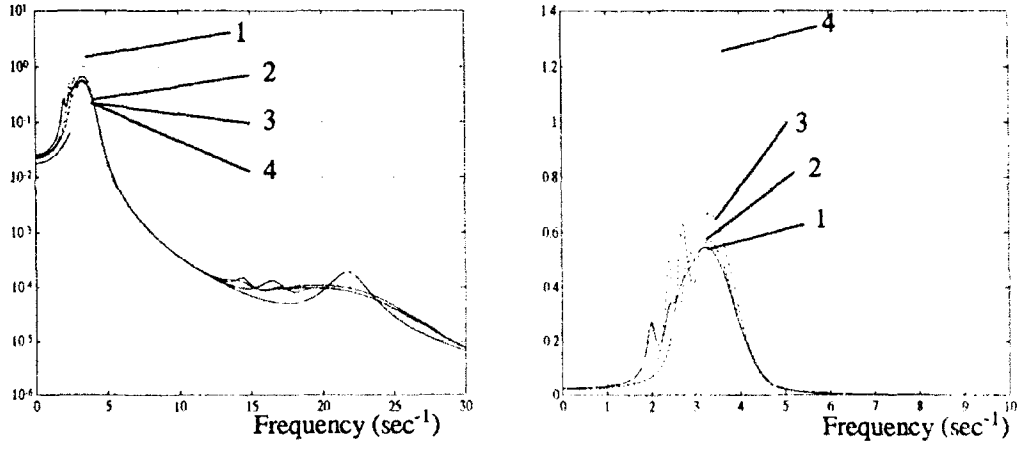


Figure 5. Spectral power density of the displacement on the top of the beam.

1) —— N = 63 (N_1=9, N_2=7) 3) - - - -N = 42 considering random damping (N_1=7,N_2=3, N_α=2)

2) - - - -N = 7 (N_1=7) 4) ..... deterministic system with mean characteristics